Divergence and Stokes' Theorem

The Punch Line: We can also compute flux and circulation by looking at boundary values in three dimensions.

For a region $W \subset \mathbb{R}^3$ with boundary surface S, and vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, we have $\iint_S \vec{F} \cdot \hat{n}\, dA = \iiint_W \operatorname{div}(\vec{F})\, dV$, where $\hat{n}$ is the outward unit normal vector, and $\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F}$ is the divergence.

For a surface $S \subset \mathbb{R}^3$ with positive boundary curve $\vec{c}$, we have $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dA$, where the orientation of $\hat{n}$ depends on the orientation of $\vec{c}$.

Computational Let W be the region in $\mathbb{R}^3$ given in polar coordinates by

$$W = \left\{(r, \theta, z) : 0 \le r \le z^4,\ 0 \le z \le 1,\ 0 \le \theta \le 2\pi\right\},$$

with boundary $S = S_1 \cup S_2$ for S_1 the lower "bowl" surface and S_2 the upper "flat disc" surface, with outward facing normals. Define $\vec{F} = (x - yz, xz - y, z - xy)$.

(a) Compute $\iint_{S_1} \vec{F} \cdot \hat{n}\, dA$.

(b) Compute $\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n}\, dA$.

(c) Compute $\iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{n}\, dA$.

Theoretical

(a) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $S(k) = \{\vec{x} : f(\vec{x}) = k\}$ denotes the level surface for value $k \in \mathbb{R}$. What is the unit normal vector to the surface $S(k)$? (There are two possible answers.)

(b) With the setup above, suppose $W(k) = \{\vec{x} : f(\vec{x}) \le k\}$ is a closed and bounded region (which then has boundary $S(k)$; this can arise with functions like $f(\vec{x}) = \|\vec{x}\|^2$). Compute $\iiint_{W(k)} \Delta f \, dV$ (where $\Delta f = \nabla \cdot \nabla f$) using the Divergence Theorem to conclude that it is nonnegative, and positive if k is not the maximum value of f.

(c) Use the Divergence Theorem to prove the following integration by parts formula for a closed and bounded region W with boundary S, and $f, g : \mathbb{R}^3 \to \mathbb{R}$ differentiable functions:

$$\iiint_W f \Delta g \, dV = \iint_S f \nabla g \cdot \hat{n} \, dA - \iiint_W \nabla f \cdot \nabla g \, dV.$$

You may need the product rule $\operatorname{div}(f\vec{F}) = \nabla f \cdot \vec{F} + f \operatorname{div}(\vec{F})$ for a differentiable scalar function f and vector field $\vec{F}$.