Sequences I

The Punch Line: We can evaluate limiting behavior of sequences by comparing them to known sequences.

A *sequence* is a function whose domain is (a subset of) the integers, usually into the reals. If $\{a_n\}$ is a sequence, it has a *limit* $L = \lim_{n\to\infty} a_n$ if for every real $\epsilon > 0$ there exists an integer $N > 0$ such that if $n \geq N$, then $|a_n - L| < \epsilon$. A sequence can fail to have a limit, in which case it is said to *diverge*.

Computational Compute the limits of the given sequences, or prove that they diverge.

(a) $a_n = \left(1 + \frac{1}{n}\right)^{\frac{1}{n}}$

(b) $a_n = \frac{\left(\frac{\sin x + \cos x}{2}\right)^n}{n+1}$ for fixed x (does your answer depend on x?)

(c) $a_n = \frac{e^{1+\frac{1}{n}}}{n} + \frac{e^{1+\frac{2}{n}}}{n} + \cdots + \frac{e^{1+\frac{n}{n}}}{n}$

(d) $a_n = \frac{n^2 \sin\left(\frac{n\pi}{2}\right)}{n^2+1}$

(a) We observe that $a_n \geq 1^{\frac{1}{n}} = 1$, so $L = \lim_{n\to\infty} a_n \geq 1$. Also, $a_n \leq 2^{\frac{1}{n}}$, so $L = \lim_{n\to\infty} a_n \leq 1$. Thus, $L = 1$.

(b) Since $\left|\frac{\sin x + \cos x}{2}\right| \leq 1$, we have that $|a_n| \leq \frac{1}{n+1}$. Thus, $\lim_{n\to\infty} |a_n| \leq 0$, so it must be equal to zero (it is composed of nonnegative terms), so $\lim_{n\to\infty} a_n = 0$ as well.

(c) Writing this as $a_n = \sum_{k=1}^{n} \frac{1}{n} e^{1+\frac{k}{n}}$, we recognize this as a right Riemann sum for the integral $\int_1^2 e^x\,dx$. This integral converges, so the Riemann sum must also converge; it then has the value $\lim_{n\to\infty} a_n = \int_1^2 e^x\,dx = e^2 - e$.

(d) We first divide both numerator and denominator by n^2 to obtain $a_n = \frac{\sin\left(\frac{n\pi}{2}\right)}{1+\frac{1}{n^2}}$. If $n = 1, 5, 9, 13, \ldots$, then $a_n = \frac{1}{1+\frac{1}{n^2}}$, so the limit of these terms is 1. If $n = 3, 7, 11, 15, \ldots$, then $a_n = \frac{-1}{1+\frac{1}{n^2}}$, so the limit of these terms is -1. Thus, a_n diverges, because for any $N > 0$, there are $n > N$ with a_n close to 1 and also $n > N$ with a_n close to -1.

Theoretical

(a) Suppose $a_{n+2} - \frac{5}{2}a_{n+1} + a_n = 0$ defines a sequence a_n, and suppose $\lim_{n\to\infty} a_n = L$ (that is, the sequence a_n converges). What must L be? Is there a sequence a_n satisfying the relation above which diverges?

(b) Suppose $\lim_{n\to\infty} a_n = L$. Prove that, given $\epsilon > 0$, there is an integer $N > 0$ such that if $n, m > N$, we have $|a_n - a_m| < \epsilon$.

(c) Suppose $\lim_{n\to\infty} a_n = L$. Does the sequence $b_n = \frac{1}{n}\sum_{k=1}^{n} a_k$ converge, and if so to what?

(a) If $\lim_{n\to\infty} a_n = L$, then L is also the limit of a_{n+1} and a_{n+2} (these are just translates of the original sequence). Taking the limit of the defining relation then gives that $L - \frac{5}{2}L + L = 0$, so $\frac{-1}{2}L = 0$, so $L = 0$ if it exists. However, if $a_n = 2^n$, then $a_{n+1} = 2(2^n)$ and $a_{n+2} = 4(2^n)$, so $a_{n+2} - \frac{5}{2}a_{n+1} + a_n = (4-5+1)2^n = 0$, and 2^n does not converge.

(b) If we are given $\epsilon > 0$, by assumption we may find $N > 0$ such that if $n, m > N$ we have $|a_n - L| < \frac{1}{2}\epsilon$ and $|a_m - L| < \frac{1}{2}\epsilon$. Then $|a_n - a_m| = |(a_n - L) - (a_m - L)| \leq |a_n - L| + |a_m - L| = \epsilon$.

(c) We know for any $\epsilon > 0$ there exists $N > 0$ such that if $n > N$, we have $|a_n - L| < \epsilon$. We claim $\lim_{n\to\infty} b_n = L$ as well. We will show $\lim_{n\to\infty} |b_n - L| = 0$, which will imply this result. We can write $|b_n - L| = \frac{1}{n}\sum_{k=1}^{n} |a_k - L|$.

Fix $\epsilon > 0$, and choose $N > 0$ such that if $n > N$, we have $|a_n - L| < \epsilon$. Let $M = \max\{|a_k - L| : 1 \leq k \leq N\}$. Then (if $n > N$) we have $|b_n - L| = \frac{1}{n}\sum_{k=1}^{N} |a_k - L| + \frac{1}{n}\sum_{k=N+1}^{n} |a_k - L| \leq \frac{M}{n} + \frac{(n-N-1)\epsilon}{n}$. Taking the limit as $n \to \infty$ gives $\lim_{n\to\infty} |b_n - L| \leq \lim_{n\to\infty} \frac{M}{n} + \lim_{n\to\infty} \frac{(n-N-1)\epsilon}{n} = 0 + \epsilon = \epsilon$. This is true for all choices of ϵ, so $\lim_{n\to\infty} |b_n - L| = 0$.