# Sequences I

**The Punch Line:** We can evaluate limiting behavior of sequences by comparing them to known sequences.

A *sequence* is a function whose domain is (a subset of) the integers, usually into the reals. If $\{a_n\}$ is a sequence, it has a *limit* $L = \lim_{n\to\infty} a_n$ if for every real $\epsilon > 0$ there exists an integer $N > 0$ such that if $n \geq N$, then $|a_n - L| < \epsilon$. A sequence can fail to have a limit, in which case it is said to *diverge*.

**Computational** Compute the limits of the given sequences, or prove that they diverge.

(a) $a_n = \left(1 + \frac{1}{n}\right)^{\frac{1}{n}}$

(b) $a_n = \frac{\left(\frac{\sin x + \cos x}{2}\right)^n}{n+1}$ for fixed $x$ (does your answer depend on $x$?)

(c) $a_n = \frac{e^{1+\frac{1}{n}}}{n} + \frac{e^{1+\frac{2}{n}}}{n} + \cdots + \frac{e^{1+\frac{n}{n}}}{n}$

(d) $a_n = \frac{n^2 \sin\left(\frac{n\pi}{2}\right)}{n^2+1}$

**Theoretical**

(a) Suppose $a_{n+2} - \frac{5}{2}a_{n+1} + a_n = 0$ defines a sequence $a_n$, and suppose $\lim_{n\to\infty} a_n = L$ (that is, the sequence $a_n$ converges). What must $L$ be? Is there a sequence $a_n$ satisfying the relation above which diverges?

(b) Suppose $\lim_{n\to\infty} a_n = L$. Prove that, given $\epsilon > 0$, there is an integer $N > 0$ such that if $n, m > N$, we have $|a_n - a_m| < \epsilon$.

(c) Suppose $\lim_{n\to\infty} a_n = L$. Does the sequence $b_n = \frac{1}{n}\sum_{k=1}^{n} a_k$ converge, and if so to what?