Series I

The Punch Line: Series are analyzed in terms of their partial sums.

A *series* is the sum of a sequence, which is really a sequence of partial sums. If the partial sums converge, we say the series converges; likewise if it diverges. Important series are *geometric series* of the form $\sum ar^n$, and *telescoping series* of the form $\sum a_{n+1} - a_n$.

Computational Compute the limits of the given series, or prove that they diverge.

(a) $\sum_{n=3}^{\infty} \left(\frac{e}{\pi}\right)^n$

(b) $\sum_{n=5}^{\infty} ne^n$

(c) $\sum_{n=1}^{\infty} \frac{1}{n^2+1} - \frac{1}{n^2+2n+2}$

(d) $\sum_{n=1}^{\infty} \frac{2^{3-2n}+2^{2-3n}}{5^{1-n}}$

(e) $\sum_{n=1}^{\infty} 2^{-n}\sin(n\pi/2)$

(f) $\sum_{n=7}^{\infty} \frac{2}{n^2-1}$

(a) This is a modified geometric series; we re-index it with $n = m+3$ to get $\sum_{m=0}^{\infty} \left(\frac{e}{\pi}\right)^{m+3}$. To put it into the notation above, we have $a = \left(\frac{e}{\pi}\right)^3$ and $r = \frac{e}{\pi}$. We know geometric series have the limit $\frac{a}{1-r}$ if $|r| < 1$, and diverge if $r > 1$. Here, $|r| < 1$, so our limit is $\frac{\left(\frac{e}{\pi}\right)^3}{1-\frac{e}{\pi}} = \frac{e^3}{\pi^2(\pi-e)}$.

(b) Each term in this sequence is bounded below by $5e^n$, so the N^{th} partial sum is bounded below by $\sum_{n=5}^{N} 5e^N = \frac{5(e^N-1)}{e-1}$. This diverges, so the given series must as well.

(c) We recognize the denominator in the second term as $(n+1)^2+1$, so this is really the telescoping sum $-\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)^2+1} - \frac{1}{n^2+1}\right)$. Thus, the N^{th} partial sum is $S_N = \frac{1}{2} - \frac{1}{(N+1)^2+1}$, so the limit of the partial sums is $\frac{1}{2}$.

(d) We simplify this to the series $\sum_{n=1}^{\infty}\left(\frac{8}{5}\left(\frac{5}{4}\right)^n + \frac{4}{5}\left(\frac{5}{8}\right)^n\right)$, which has the divergent part with base $\frac{5}{4} > 1$, hence diverges.

(e) Here, we have several cases: if n is even, then $\sin(n\pi/2) = 0$. If $n = 1+4k$, then $\sin(n\pi/2) = 1$, and if $n = 3+4k$ then $\sin(n\pi/2) = -1$. So, we can re-write our series as $\sum_{k=0}^{\infty} 2^{-1-4k} - \sum_{k=0}^{\infty} 2^{-3-4k} = \sum_{k=0}^{\infty}(2^{-1} - 2^{-3})2^{-4k}$. This is $\frac{3}{8}\sum_{k=0}^{\infty} 16^{-k} = \frac{3}{8}\frac{1}{1-16^{-1}} = \frac{2}{5}$.

(f) We write $\frac{2}{n^2-1} = \frac{2}{(n+1)(n-1)} = \frac{1}{n-1} - \frac{1}{n+1}$. Examining the even terms $n = 2k$ gives terms $\frac{1}{2k-1} - \frac{1}{2k+1} = \frac{-1}{2(k+1)-1} - \frac{-1}{2k-1}$, and examining the odd terms $n = 2k+1$ gives terms $\frac{1}{2k} - \frac{1}{2k+2} = \frac{-1}{2(k+1)} - \frac{-1}{2k}$. Thus, our series is the sum of the two telescoping series of even and odd terms, $\sum_{n=7}^{\infty} \frac{2}{n^2-1} = \sum_{k=4}^{\infty}\left(\frac{-1}{2(k+1)-1} - \frac{-1}{2k-1}\right) + \sum_{k=3}^{\infty}\left(\frac{-1}{2(k+1)} - \frac{-1}{2k}\right) = \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$.

Theoretical

(a) Suppose $\sum_{n=0}^{\infty} a_n$ converges. Does the series $\sum_{n=N}^{\infty} a_n$ always converge for fixed $N > 0$? What can we say about its limit when it does?

(b) Suppose $\sum_{n=0}^{\infty} a_n$ converges. Does the sequence $R_N = \sum_{n=N}^{\infty} a_n$ always converge? What can we say about its limit when it does?

(c) Suppose a_n is a sequence of positive real numbers. Can $\sum_{N=0}^{\infty} \sum_{n=0}^{N} a_n$ converge?

(a) Yes—the partial sums of the series are the partial sums of the original series minus the first $N-1$ terms. We know the partial sums converge, so the difference between the partial sums and a constant sequence also converges, to the difference between the limit and that constant. So, $\sum_{n=N}^{\infty} a_n = \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{N-1} a_n$.

(b) Yes—put $S_N = \sum_{n=0}^{N} a_n$ for the partial sums, and $S = \lim_{N\to\infty} S_N$ for the limit of the convergent series. Then $R_N = S - S_N$. By the definition of limit, for every $\epsilon > 0$ there is an $M > 0$ such that if $N > M$, we have $|S - S_N| = |R_N - 0| < \epsilon$. Thus, $\lim_{N\to\infty} R_N = 0$.

(c) No—since each $a_n > 0$, we have the partial sums $S_N = \sum_{n=0}^{N} a_n$ are increasing, as $S_{N+1} - S_N = a_{N+1} > 0$. Then the M^{th} partial sum of the overall series $\sum_{N=0}^{M} S_N > \sum_{N=0}^{M} S_0 = MS_0$, so the limit is $\lim_{M\to\infty} MS_0 = S_0 \lim_{M\to\infty} M$, which diverges to infinity (noting that $S_0 = a_0 > 0$).