Series II

The Punch Line: We can apply various tests to attempt to determine convergence. Important ones are the ratio test (examining $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$) and the root test (examining $\lim_{n\to\infty} \sqrt[n]{a_n}$) for series with positive terms, and the integral test (examining $\int_N^\infty f\,dx$) for series with positive decreasing terms.

Computational Compute the limits of the given series, or prove that they diverge.

(a) $\sum_{n=0}^{\infty} \frac{(2n)!}{(2n)^n}$

(b) $\sum_{n=1}^{\infty} n^{-2n}$

(c) $\sum_{n=1}^{\infty} \frac{1+2n}{(n+1)^\pi}$

(d) $\sum_{n=2}^{\infty} \frac{\sin(n)}{n^2 \ln n}$

(a) We use the ratio test, computing

$$\lim_{n\to\infty} \frac{\frac{(2n+2)!}{(2n+2)^{n+1}}}{\frac{(2n)!}{(2n)^n}} = \lim_{n\to\infty} \frac{2^n n^n}{2^n(2n+2)(n+1)^n} \frac{(2n+2)(2n+1)(2n)!}{(2n)!} = \lim_{n\to\infty} (2n+1)\left(\frac{n}{n+1}\right)^n = \infty,$$

so the series diverges.

(b) We use the root test, computing

$$\lim_{n\to\infty} \sqrt[n]{n^{-2n}} = \lim_{n\to\infty} n^{-2} = 0,$$

so the series converges.

(c) We write this as $\sum_{n=1}^{\infty} \frac{-1}{(n+1)^\pi} + \frac{2(n+1)}{(n+1)^\pi}$. Now, we could re-index with $m = n+1$ to obtain a p-series with $p = \pi$. We can also directly use the integral test, examining

$$\int_1^\infty \frac{2}{(n+1)^{\pi-1}} - \frac{1}{(n+1)^\pi}\,dx = \left[\frac{2(2-\pi)}{(n+1)^{\pi-2}} + \frac{\pi}{(n+1)^{\pi-1}}\right]_1^\infty = \frac{2(2-\pi)}{2^{\pi-2}} + \frac{\pi}{2^{\pi-1}},$$

which is finite, so the series converges.

(d) We examine the series whose terms are the absolute values of the given series. This is $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$. The denominator satisfies $n^2 \ln n > n^2$ for $n > 2$ (not for $n = 2$, but that won't harm convergence), so the terms satisfy $\frac{1}{n^2 \ln n} < \frac{1}{n^2}$, and the latter forms a convergent series. Thus, the given series is convergent.

Challenging

(a) $\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n-1} k^{-2}}{\sum_{k=n}^{\infty} k^{-2}}$

(b) $\sum_{n=2}^{\infty} \sum_{m=n}^{\infty} \left(\frac{1}{n}\right)^m$

(a) The numerator is always bounded below by 1. The denominator consists of the tails of a convergent series, and so forms a sequence b_n converging to zero (from above). Thus, the sequence of terms is bounded below by b_n^{-1}, a sequence which diverges to infinity. Thus the series diverges to infinity.

(b) For any fixed n, we have $\sum_{m=n}^{\infty} \left(\frac{1}{n}\right)^m = \frac{n^{-n}}{1-\frac{1}{n}} = \frac{n^{1-n}}{n-1}$. Then we are left with $\sum_{n=2}^{\infty} \frac{n^{1-n}}{n-1}$. The terms are bounded above by $n^{1-n} < n2^{-n}$. Applying the ratio test to this, we examine $\lim_{n\to\infty} \frac{(n+1)2^{-n-1}}{n2^{-n}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$. Thus, it converges.