Taylor Series

The Punch Line: Differentiable functions admit optimal polynomial approximations around specified points, which are easily computed in terms of the derivatives. The *Taylor series* of the smooth function $f(x)$ about the point x_0 is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

If $x_0 = 0$, this is called a *Maclaurin series*, and the partial sums are called the *Taylor* (or *Maclaurin*) *polynomials* of the appropriate order.

Computational Compute the Taylor polynomials of the specified orders about the appropriate points of the following functions:

(a) Order 2; $f(x) = e^{-2x}$ about $x_0 = \ln 2$

(b) Order 4; $f(x) = 2\sin(2x)$ about $x_0 = \pi$

(c) Order 3; $f(x) = \sin(x^2)$ about $x_0 = 0$

(d) Order 2; $f(x) = e^{\sin x}$ about $x_0 = 0$

(e) Order 3; $f(x) = (1+x)^n$ about $x_0 = 0$

(f) Order 1; $f(x) = |x|$ about any $x_0 \neq 0$

(a) We have $f'(x) = -2e^{-2x}$ and $f''(x) = 4e^{-2x}$, so we have the Taylor polynomial $p_2(x) = \frac{1}{4} - \frac{1}{2}(x - \ln 2) + \frac{1}{2}(x - \ln 2)^2$.

(b) We have $f'(x) = 4\cos(2x)$, $f''(x) = -8\sin(2x)$, $f'''(x) = -16\cos(2x)$, and $f^{(4)}(x) = 32\sin(2x)$. Thus, the Taylor polynomial $p_4(x) = 4(x-\pi) - \frac{8}{3}(x-\pi)^3$.

(c) We have $f'(x) = 2x\cos(x^2)$, $f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$, and $f'''(x) = -4x\sin(x^2) - 8x\sin(x^2) - 8x^3\cos(x^2)$. Then $p_3(x) = x^2$.

(d) We have $f'(x) = \cos x e^{\sin x}$ and $f''(x) = -\sin x e^{\sin x} + \cos^2 x e^{\sin x}$, so $p_2(x) = 1 + x + \frac{1}{2}x^2$. It's worth noting that this is indistinguishable from e^x at this point; the functions only begin to differ at the third order.

(e) We have $f'(x) = n(1+x)^{n-1}$, $f''(x) = n(n-1)(1+x)^{n-2}$, and $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$ (with the understanding that the k^{th} derivative is zero if $k > n$). Then $p_3(x) = 1 + nx + \frac{1}{2}n(n-1)x^2 + \frac{1}{6}n(n-1)(n-2)x^3$.

(f) We have $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x < 0$ (f is not differentiable at zero). Thus, if $x_0 < 0$, we have $p_1(x) = -x_0 - (x - x_0)$, while if $x_0 > 0$, we have $p_1(x) = x_0 + (x - x_0)$. While we usually write Taylor series in terms of powers of $(x - x_0)$, it is worth mentioning that in the former case, we have $p_1(x) = -x$, while in the latter $p_1(x) = x$; that is, the Taylor series only "knows about" the branch of the absolute value function it is on, not about the turn.

Theoretical

(a) Suppose

$$f(x) = \begin{cases} e^{-x^{-2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

What is the Maclaurin series for f?

[This is quite difficult, but it is worth investigating the first few Maclaurin polynomials even without a general result.]

(b) If $p(x)$ is a polynomial, show that the Taylor coefficients of p about x_0 are polynomials in x_0. Show this explicitly for $p(x) = x^3 - x$.

(c) Suppose $f'(x) = xf(x)$ and $f(0) = 1$. Solve this differential equation by considering a Maclaurin series for f and solving for the coefficients, then showing this converges for all x.

(a) We'll show that $f^{(k)} = \frac{P_k(x)}{x^{3k}} f(x)$ where P_k is a polynomial of degree $2(k-1)$ by using a technique called mathematical induction (if this is unfamiliar, this argument may seem very strange, but it is valid). The first derivative is $f'(x) = \frac{2}{x^3} f(x)$, so the claim is true for $k = 1$. Suppose $f^{(k-1)}(x) = \frac{P_{k-1}(x)}{x^{3k-3}} f(x)$ for some polynomial P_{k-1} of degree $2(k-2)$. Then

$$f^{(k)}(x) = \left(\frac{P'_{k-1}(x)}{x^{3k-3}} - (3k-3)\frac{P_{k-1}(x)}{x^{3k-2}} + \frac{2P_{k-1}(x)}{x^{3k}} \right) f(x) = \frac{x^3 P'_{k-1}(x) - (3k-3)x^2 P_{k-1}(x) + 2P_{k-1}(x)}{x^{3k}} f(x),$$

and we can see that the numerator has gone up in degree by two, so the claim is true of k as well. Since $e^{-x^{-2}}$ tends to zero faster than any polynomial (this requires some checking, but is true), the derivative $f^{(k)}(0)$ is then 0 for all k. Thus, the Maclaurin series for f is the zero function.

(b) The derivative of a polynomial is a polynomial of one lower degree, so all derivatives of $p(x)$ are polynomials. The Taylor coefficients are multiples of the derivatives, so this is enough to say that they are polynomials. For $p(x) = x^3 - 1$, we have Taylor coefficients $x_0^3 - x_0$, $3x_0^2 - 1$, $3x_0$, and 1.

(c) Suppose $f(x) = \sum_{k=0}^{\infty} c_k x^k$. Then $f'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k$, while $x f(x) = \sum_{k=0}^{\infty} c_k x^{k+1} = \sum_{k=1}^{\infty} c_{k-1} x^k$. Thus, our differential equation requires

$$c_1 + \sum_{k=1}^{\infty} (k+1) c_{k+1} x^k = \sum_{k=1}^{\infty} c_{k-1} x^k.$$

Equating like terms, we see that $c_1 = 0$ and $(k+1)c_{k+1} = c_{k-1}$ for $k \geq 1$. Re-writing this, $c_{k+2} = \frac{c_k}{k+2}$ for all k. In particular, this means that for k odd, $c_k = 0$, and if $k = 2j$, we have $c_{2(j+1)} = \frac{1}{(2j+2)!!}$ (where $n!! = n(n-2)(n-4)\cdots$, stepping down by powers of two) because $c_0 = f(0) = 1$. Thus, $f(x) = \sum_{j=0}^{\infty} \frac{1}{(2j)!!} x^{2j}$.

Letting $a_j = \frac{x^{2j}}{(2j)!!}$, we consider $\frac{a_{j+1}}{a_j} = \frac{x^{2j} x^2 (2j)!!}{(2j+2)(2j)!! x^{2j}} = \frac{x^2}{2j+2}$. For any fixed x, as $j \to \infty$ this converges to zero, so by the Ratio Test $f(x)$ is defined for all x. In fact, $f(x) = e^{x^2/2}$, as simple separation of variables can show, but this technique can solve differential equations which are not tractable otherwise.