Taylor Series

The Punch Line: Differentiable functions admit optimal polynomial approximations around specified points, which are easily computed in terms of the derivatives. The *Taylor series* of the smooth function $f(x)$ about the point x_0 is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

If $x_0 = 0$, this is called a *Maclaurin series*, and the partial sums are called the *Taylor* (or *Maclaurin*) *polynomials* of the appropriate order.

Computational Compute the Taylor polynomials of the specified orders about the appropriate points of the following functions:

(a) Order 2; $f(x) = e^{-2x}$ about $x_0 = \ln 2$

(b) Order 4; $f(x) = 2\sin(2x)$ about $x_0 = \pi$

(c) Order 3; $f(x) = \sin(x^2)$ about $x_0 = 0$

(d) Order 2; $f(x) = e^{\sin x}$ about $x_0 = 0$

(e) Order 3; $f(x) = (1 + x)^n$ about $x_0 = 0$

(f) Order 1; $f(x) = |x|$ about any $x_0 \neq 0$

Theoretical

(a) Suppose

$$f(x) = \begin{cases} e^{-x^{-2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

What is the Maclaurin series for f?

[This is quite difficult, but it is worth investigating the first few Maclaurin polynomials even without a general result.]

(b) If $p(x)$ is a polynomial, show that the Taylor coefficients of p about x_0 are polynomials in x_0. Show this explicitly for $p(x) = x^3 - x$.

(c) Suppose $f'(x) = xf(x)$ and $f(0) = 1$. Solve this differential equation by considering a Maclaurin series for f and solving for the coefficients, then showing this converges for all x.