Power Series

The Punch Line: We can manipulate convergent power series to learn about the functions they represent, and vice versa.

If $f(x) = \sum_{k=0}^{\infty} c_k x^k$ on the interval $(-r, r)$, then $f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}\left[c_k x^k\right]$ and $\int f\,dx = C + \sum_{k=0}^{\infty} \int c_k x^k\,dx$, and algebraic operations between power series are valid on (at least) the intersection of their radii of convergence.

Computational

(a) What is the power series for $\sin\left(\pi x^2\right)$? What is its radius of convergence?

(b) What is the function represented by

$$\sum_{k=0}^{\infty}(k+1)(k+2)x^k?$$

What is an interval on which this is valid?

(c) What is a Taylor series for $\frac{1}{1-x}$ about the point $x_0 = 5$? What is the radius of convergence here?

(d) Let $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}$. What is the radius of convergence of this series? Give a series representation of $f'(x)$.

(a) We know that $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, so

$$\sin(\pi x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\pi x^2\right)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1} x^{4k+2}}{(2k+1)!}.$$

Both $\sin x$ and πx^2 have infinite radii of convergence, so so too does $\sin(\pi x^2)$.

(b) Reindexing so $j = k-2$, we have $\sum_{j=2}^{\infty} j(j-1)x^{j-2}$. We can see that the second antiderivative of this (termwise) is $C_0 + C_1 x + \sum_{j=2}^{\infty} x^j = C_0 + C_1 x + \frac{1}{1-x}$, so we have the second derivative $\frac{d^2}{dx^2}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] = \frac{2}{(1-x)^3}$. As the series for $\frac{1}{1-x}$ is valid when $|x| < 1$, so too is this series.

(c) A series for x about $x_0 = 2$ is a series for $y = x-5$ about $y_0 = 0$; so, we make the substitution $x = y+5$ to obtain the function $\frac{1}{1-y-5} = \frac{-1/4}{1-\left(\frac{-y}{4}\right)}$. For $\left|-y\right| = \left|y\right| < 4$, this has the series representation $-\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{-y}{4}\right)^k$, so for $|x-5| < 4$ we have the Taylor series $\frac{1}{1-x} = -\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{5-x}{4}\right)^k = \sum_{k=0}^{\infty}\left(\frac{-1}{4}\right)^{k+1}(x-5)^k$.

(d) We see that for all x, we have $\left|\frac{x^k}{(k!)^2}\right| \leq \left|\frac{x^k}{k!}\right|$, which are the terms of the power series for $e^{|x|}$. Thus, the given series converges everywhere that function does, and it converges everywhere. Thus, we have an infinite radius of convergence. The derivative has the power series $\sum_{k=1}^{\infty} \frac{x^{k-1}}{k!(k-1)!}$ (note that the summation is from 1 rather than zero: the derivative of the constant term is zero, rather than proportional to x^{-1}).

Theoretical

(a) Show that the power series $\sum_{k=0}^{\infty} c_k x^k$ satisfying $c_{k+2} = \frac{1}{k+2} c_k$ all converge, and form a two-dimensional vector space.

(b) Show that $\sum_{k=0}^{\infty} x^{-k}$ converges for $|x| > M$ for some M (find it), and to which function.

(a) We can see that for $k = 2j$, we have $c_k = \frac{1}{(2j)!!} c_0 = \frac{1}{(2j)(2j-2)\cdots(2)} c_0 = \frac{2^j}{j!} c_0$. Similarly, for $k = 2j+1$, we have $c_k = \frac{1}{(2j+1)!!} c_0 = \frac{1}{(2j+1)(2j-1)\cdots(3)(1)} c_1$. Thus, the series of even and odd terms both converge (we have previously shown this; it is also fairly easy to compare both to convergent p-series), so the overall series converges. We are free to choose c_0 and c_1, so it is clear that all linear combinations of such series are allowed, giving us the vector space structure.

(b) We set $y = \frac{1}{x}$; then the series is $\sum_{k=0}^{\infty} y^k = \frac{1}{1-y}$ for $\left|y\right| < 1$. This happens when $|x| > 1$, and the limit is then $\frac{1}{1-\frac{1}{x}} = \frac{x}{x-1}$. This is an example of a "Taylor Series at infinity", which is sometimes a helpful concept.