Power Series

The Punch Line: We can manipulate convergent power series to learn about the functions they represent, and vice versa.

If $f(x) = \sum_{k=0}^{\infty} c_k x^k$ on the interval $(-r, r)$, then $f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}\left[c_k x^k\right]$ and $\int f\,dx = C + \sum_{k=0}^{\infty} \int c_k x^k\,dx$, and algebraic operations between power series are valid on (at least) the intersection of their radii of convergence.

Computational

(a) What is the power series for $\sin\left(\pi x^2\right)$? What is its radius of convergence?

(b) What is the function represented by

$$\sum_{k=0}^{\infty}(k+1)(k+2)x^k?$$

What is an interval on which this is valid?

(c) What is a Taylor series for $\frac{1}{1-x}$ about the point $x_0 = 5$? What is the radius of convergence here?

(d) Let $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}$. What is the radius of convergence of this series? Give a series representation of $f'(x)$.

Theoretical

(a) Show that the power series $\sum_{k=0}^{\infty} c_k x^k$ satisfying $c_{k+2} = \frac{1}{k+2} c_k$ all converge, and form a two-dimensional vector space.

(b) Show that $\sum_{k=0}^{\infty} x^{-k}$ converges for $|x| > M$ for some M (find it), and to which function.