Fourier Series

The Punch Line: For periodic functions, we can expand in terms of trigonometric functions with the same period.

If $f(x)$ is periodic in the sense that $f(x+2L) = f(x)$, then we can examine $f(x)$ on the interval $[-L, L]$ to obtain an expansion in terms of sine and cosine functions:

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right),$$

which is valid wherever f is sufficiently smooth. In particular, if f has left and right derivatives at all points in $[-L, L]$, the *Fourier series* above converges at all x at which f is continuous. If f is an even function, only the a_k are nonzero, while if it is odd, only the b_k are nonzero.

Computational Compute Fourier series for the following functions, on the appropriate intervals.

(a)
$$f(x) = x \text{ on } [-\pi, \pi]$$

(b)
$$f(x) = x \text{ on } [0, 1]$$

(this is the "fractional part" of x)

(c)
$$f(x) = \begin{cases} 1 + x, & -1 \le x \le 0, \\ 1 - x, & 0 \le x \le 1 \end{cases}$$

(this is a "triangular wave")

(d) The odd extension of
$$f(x) = x^2 \text{ on } [0, 1]$$

(a) We see that this is an odd function, so we only need the sine series. We compute

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx = \frac{1}{\pi}\left[-\frac{1}{k}x\cos(kx)\right]_{-\pi}^{\pi} + \frac{1}{k\pi}\int_{-\pi}^{\pi}\cos(kx)\,dx = \frac{2}{k}(-1)^{k+1}.$$

Thus, the Fourier series for f is

$$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{2}{k}\sin(kx).$$

This converges for $x \ne n\pi$, $n \in \mathbb{Z}$.

(b) Here, we're periodic about $\frac{1}{2}$. We could compute on the interval $[-1/2, 1/2]$, but it'll actually be easier to reparametrize and use the above result. Let $y = 2\pi x - \pi$; we examine the function $x = \frac{y+\pi}{2\pi}$ on $y \in [-\pi, \pi]$. By the above considerations, we have the Fourier series

$$\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\pi}\sin(ky) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k\pi}\sin(2\pi kx - k\pi) = \frac{1}{2} - \sum_{k=1}^{\infty}\frac{1}{k\pi}\sin(2\pi kx),$$

where we have used the fact that shifting sine by π negates it.

(c) This is an even function, so we only need the cosine series. The average value is $a_0 = \frac{1}{2}$ (consider the area of the triangle). We compute for $k > 0$

$$\begin{aligned}
a_k =& \int_{-1}^{0} (1+x)\cos(k\pi x)\,dx + \int_{0}^{1} (1-x)\cos(k\pi x), dx \\
=& \left[\frac{1}{k\pi} x \sin(k\pi x)\right]_{-1}^{0} - \left[\frac{1}{k\pi} x \sin(k\pi x)\right]_{0}^{1} + \int_{-1}^{0} \frac{1}{k\pi}\sin(k\pi x)\,dx - \int_{0}^{1} \frac{1}{k\pi}\sin(k\pi x)\,dx + \int_{-1}^{1} \cos(k\pi x)\,dx \\
=& 0 - 0 + \frac{1}{k^2\pi^2}\left[-\cos(k\pi x)\right]_{-1}^{0} - \frac{1}{k^2\pi^2}\left[-\cos(k\pi x)\right]_{0}^{1} + 0 \\
=& \frac{2(1-\cos(k\pi))}{k^2\pi^2} = \begin{cases} \frac{4}{k^2\pi^2}, & k = 2j+1, \\ 0, & k = 2j \end{cases}
\end{aligned}$$

Thus, our Fourier series is

$$\frac{1}{2} + \sum_{j=0}^{\infty} \frac{4}{(2j+1)^2\pi^2} \cos\left((2j+1)\pi x\right).$$

(d) To take the odd extension, we consider the sine series. We compute

$$\begin{aligned}
b_k =& 2\int_{0}^{1} x^2 \sin(k\pi x)\,dx \\
=& \left[\frac{-2x^2}{k\pi}\cos(k\pi x)\right]_{0}^{1} + 4\int_{0}^{1} \frac{x}{k\pi}\cos(k\pi x)\,dx \\
=& \frac{(-1)^{k+1}2}{k\pi} + 4\left[\frac{x}{k^2\pi^2}\sin(k\pi x)\right]_{0}^{1} - 4\int_{0}^{1} \frac{1}{k^2\pi^2}\sin(k\pi x)\,dx \\
=& \frac{(-1)^{k+1}2}{k\pi} + 4\left[\frac{1}{k^3\pi^3}\cos(k\pi x)\right]_{0}^{1} \\
=& \frac{(-1)^{k+1}2k^2\pi^2 + 4(-1)^k - 4}{k^3\pi^3} = \frac{2(-1)^k\left(2-k^2\pi^2\right)-4}{k^3\pi^3}.
\end{aligned}$$

So, our Fourier series is

$$\sum_{k=1}^{\infty} \frac{2(-1)^k(2-\pi^2k^2)-4}{\pi^3k^3}\sin(\pi k x).$$

Theoretical

(a) What happens to the Fourier series for the function $f(x) = x$ on $[-L, L]$ as L converges to zero? Does this make sense?

(b) What conditions on $f(x)$, if any, are necessary for $F(x) = \int_{-\pi}^{x} f(x)\,dx$ to be periodic with period 2π?

(c) If $f(x)$ has a Fourier series on $[-\pi, \pi]$ with even coefficients a_k and odd coefficients b_k, what is the Fourier series for $f'(x)$? (Prove this without differentiating termwise.)

(a) We only need the sine series, and setting $x = \frac{L}{\pi}y$ we compute

$$b_k = \frac{1}{L}\int_{-L}^{L} x\sin(\frac{k\pi}{L}x)\,dx = \frac{L}{\pi^2}\int_{-\pi}^{\pi} y\sin(ky)\,dy = \frac{2L}{k\pi}(-1)^{k+1}.$$

As L tends to zero, the sine terms are bounded, and the coefficients decreasing to zero. Thus, the Fourier series tends to the zero series. This makes sense, as on $[-L, L]$, $|f(x)| \le L$: with a small period, the function doesn't grow large, and in fact tends pointwise to zero.

(b) We need f to be integrable (of course) and periodic, and then there is simply the requirement that $F(\pi) = F(-\pi)$, which requires that $\int_{-\pi}^{\pi} f(x)\,dx = 0$.

(c) We see that $f'(x)$ must have mean zero (else its antiderivative would accumulate value between periods). We compute

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos(kx)\,dx =& \frac{1}{\pi}\left[kf(x)\sin(kx)\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} kf(x)\sin(kx)\,dx \\
=& -kb_k, \\
\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(kx)\,dx =& \frac{1}{\pi}\left[-kf(x)\cos(kx)\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{\pi}^{\pi} kf(x)\cos(kx)\,dx \\
=& ka_k.
\end{aligned}$$

Thus, the Fourier series for $f'(x)$ is

$$\sum_{k=1}^{\infty} ka_k\sin(kx) - kb_k\cos(kx).$$