Fourier Series

The Punch Line: For periodic functions, we can expand in terms of trigonometric functions with the same period.

If $f(x)$ is periodic in the sense that $f(x+2L) = f(x)$, then we can examine $f(x)$ on the interval $[-L, L]$ to obtain an expansion in terms of sine and cosine functions:

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right),$$

which is valid wherever f is sufficiently smooth. In particular, if f has left and right derivatives at all points in $[-L, L]$, the *Fourier series* above converges at all x at which f is continuous. If f is an even function, only the a_k are nonzero, while if it is odd, only the b_k are nonzero.

Computational Compute Fourier series for the following functions, on the appropriate intervals.

(a)

$$f(x) = x \text{ on } [-\pi, \pi]$$

(b)

$$f(x) = x \text{ on } [0, 1]$$

(this is the "fractional part" of x)

(c)

$$f(x) = \begin{cases} 1 + x, & -1 \le x \le 0, \\ 1 - x, & 0 \le x \le 1 \end{cases}$$

(this is a "triangular wave")

(d) The odd extension of

$$f(x) = x^2 \text{ on } [0, 1]$$

Theoretical

(a) What happens to the Fourier series for the function $f(x) = x$ on $[-L, L]$ as L converges to zero? Does this make sense?

(b) What conditions on $f(x)$, if any, are necessary for $F(x) = \int_{-\pi}^{x} f(x)\,dx$ to be periodic with period 2π?

(c) If $f(x)$ has a Fourier series on $[-\pi, \pi]$ with even coefficients a_k and odd coefficients b_k, what is the Fourier series for $f'(x)$? (Prove this without differentiating termwise.)