Fourier Transform

The Punch Line: For absolutely integrable functions on $\mathbb{R}$, we can define the *Fourier Transform*. This essentially gives the coefficients necessary to represent the function as a "sum" (integral) of complex exponentials (analogous to trigonometric functions). Intuitively, we are converting from a spatial variable x to a frequency variable ω.

Computational Compute Fourier transforms for the following functions.

(a)
$$f(x) = \begin{cases} \frac{n}{2}, & x \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\ 0, & \text{else} \end{cases}$$

(b)
$$f(x) = \begin{cases} e^{i\omega_0 x}, & |x| < \frac{n\pi}{\omega_0}, \\ 0, & |x| \geq \frac{n\pi}{\omega_0} \end{cases}$$

(c)
$$f(x) = e^{-|x|}$$

(d)
$$f(x) = \begin{cases} 2, & x \in [-1, 1], \\ 1, & x \in [-2, -1] \cup [1, 2], \\ 0, & \text{else} \end{cases}$$

(a) We compute
$$F[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2} e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{n}{-2i\omega} e^{-i\omega x} \right]_{-\frac{1}{n}}^{\frac{1}{n}} = \frac{1}{\sqrt{2\pi}} \frac{\sin\left(\frac{\omega}{n}\right)}{\frac{\omega}{n}}.$$

(b) We compute
$$F[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{n\pi}{\omega_0}}^{\frac{n\pi}{\omega_0}} e^{-i\omega x} e^{i\omega_0 x}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{i(\omega_0 - \omega)} e^{i(\omega_0 - \omega)x} \right]_{-\frac{n\pi}{\omega_0}}^{\frac{n\pi}{\omega}} = \frac{1}{\sqrt{2\pi}} \frac{2}{\omega - \omega_0} \sin\left(n\pi \frac{\omega}{\omega_0}\right).$$

(c) We compute
$$\begin{aligned} F[f](\omega) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-i\omega x} e^{x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-i\omega x} e^{-x}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1 - i\omega} e^{(1-i\omega)x} \right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{1 + i\omega} e^{-(1+i\omega)x} \right]_{0}^{\infty} \\ =& \frac{1}{\sqrt{2\pi}} \frac{1}{1 - i\omega} + \frac{1}{\sqrt{2\pi}} \frac{1}{1 + i\omega} = \frac{1}{\sqrt{2\pi}} \frac{2}{1 + \omega^2}. \end{aligned}$$

(d) We compute
$$F[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} e^{-i\omega x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-i\omega} e^{-i\omega x} \right]_{-2}^{2} + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-i\omega} e^{-i\omega x} \right]_{-1}^{1} = \frac{2}{\sqrt{2\pi}} \left(\frac{\sin(2\omega)}{\omega} + \frac{\sin(\omega)}{\omega} \right).$$

Theoretical

(a) Suppose u solves the differential equation $u'' + 2u' + 2u = e^{-|x|}$. What is $F[u](\omega)$?

(b) Without computing directly, what is the Fourier transform of $f(x) = \int_0^x e^{-|t|}\,dt$?

(c) Suppose u and v have Fourier transforms $\hat{u}$ and $\hat{v}$, respectively. Define the function

$$w(x) = \int_{-\infty}^{\infty} u(x-y)v(y)\,dy$$

(this is the *convolution* of u and v). What is $F[w](\omega)$?

(d) Use the above two parts to deduce a general integral formula for the solution u of the differential equation $-u''(x) + u(x) = g(x)$, assuming g is absolutely integrable.

(a) We know that the Fourier transform of the derivatives of u can be expressed as multiples by $i\omega$ of the Fourier transform of u. Thus, taking the Fourier transform on both sides of the equation gives that

$$(-\omega^2 + 2i\omega + 2)F[u](\omega) = \frac{1}{\sqrt{2\pi}}\frac{2}{1+\omega^2}.$$

Then we simply solve algebraically to get

$$F[u](\omega) = \frac{1}{\sqrt{2\pi}}\frac{2}{(1+\omega^2)(2+2i\omega-2)}.$$

(b) We know that $f'(x) = e^{-|x|}$, and therefore $F[f'](\omega) = i\omega F[f](\omega) = \frac{1}{\sqrt{2\pi}}\frac{2}{1+\omega^2}$, so $F[f](\omega) = \frac{1}{\sqrt{2\pi}}\frac{-2i}{\omega(1+\omega^2)}$.

(c) We compute

$$\begin{aligned}
F[w](\omega) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega x}u(x-y)v(y)\,dy\,dx \\
=& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega(x-y)}u(x-y)e^{-i\omega y}v(y)\,dy\,dx \\
=& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega(x-y)}u(x-y)e^{-i\omega y}v(y)\,dx\,dy \\
=& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega z}u(z)e^{-i\omega y}v(y)\,dz\,dy \\
=& \sqrt{2\pi}\,\hat{u}(\omega)\hat{v}(\omega).
\end{aligned}$$

Note that being able to switch the bounds of integration here was absolutely crucial. This is the fascinating statement that the Fourier transform changes convolution in space to multiplication of frequency.

(d) We take the Fourier transform of both sides of the equation, yielding the frequency-domain equation

$$(\omega^2 + 1)F[u](\omega) = F[g](\omega).$$

Algebraically solving then gives $F[u](\omega) = \frac{F[g](\omega)}{1+\omega^2} = \frac{1}{2}F[g](\omega)\left(\frac{2}{1+\omega^2}\right)$. Using the previous result, this multiplication of Fourier transforms gives the convolution of the original functions, so

$$u(x) = \frac{1}{2}\int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} e^{-|x-y|}g(y)\,dy\,dt.$$

Note that, on the whole space, there is a unique solution to the differential equation (we do not need to specify initial data like we do on half-spaces).