Fourier Transform

The Punch Line: For absolutely integrable functions on $\mathbb{R}$, we can define the *Fourier Transform*. This essentially gives the coefficients necessary to represent the function as a "sum" (integral) of complex exponentials (analogous to trigonometric functions). Intuitively, we are converting from a spatial variable x to a frequency variable ω.

Computational Compute Fourier transforms for the following functions.

(a)

$$f(x) = \begin{cases} \frac{n}{2}, & x \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\ 0, & \text{else} \end{cases}$$

(b)

$$f(x) = \begin{cases} e^{i\omega_0 x}, & |x| < \frac{n\pi}{\omega_0}, \\ 0, & |x| \geq \frac{n\pi}{\omega_0} \end{cases}$$

(c)

$$f(x) = e^{-|x|}$$

(d)

$$f(x) = \begin{cases} 2, & x \in [-1, 1], \\ 1, & x \in [-2, -1] \cup [1, 2], \\ 0, & \text{else} \end{cases}$$

Theoretical

(a) Suppose u solves the differential equation $u'' + 2u' + 2u = e^{-|x|}$. What is $F[u](\omega)$?

(b) Without computing directly, what is the Fourier transform of $f(x) = \int_0^x e^{-|t|}\,dt$?

(c) Suppose u and v have Fourier transforms $\hat{u}$ and $\hat{v}$, respectively. Define the function

$$w(x) = \int_{-\infty}^{\infty} u(x-y)v(y)\,dy$$

(this is the *convolution* of u and v). What is $F[w](\omega)$?

(d) Use the above two parts to deduce a general integral formula for the solution u of the differential equation $-u''(x) + u(x) = g(x)$, assuming g is absolutely integrable.