PDEs

The Punch Line: For *separable* partial differential equations, we can write solutions as sums of product solutions (that is, for two dimensional solutions, writing $u(x,t) = G(x)T(t)$). If the differential equation is suitable, this allows us to write ordinary differential equations for the different factors in terms of various parameters, which we can solve to find an overall solution. Various integrals allow us to match the initial conditions.

Computational Solve the following initial value-boundary problems:

(a) On the region $0 < x < 1$, the wave equation

$$u_{tt} = 25u_{xx}$$

with zero boundary conditions, initial condition $u(x,0) = x(1-x)$, and initial derivative $u_t(x,0) = \sin(\pi x)$.

(b) On the region $0 < x < \pi$, the heat equation

$$u_t = 4u_{xx}$$

with zero boundary conditions, and initial condition $u(x,0) = \sin(x) + \sin(3x)$.

(a) We first explore solutions of the form $G(x)T(t)$; for these, the differential equation yields

$$\frac{T''}{25T} = \frac{G''}{G} = -k,$$

for some constant k (it will be convenient to use this sign convention later). Then we have the system

$$\begin{cases} T''(t) + 25kT(t) = 0, \\ G''(x) + kG(x) = 0. \end{cases}$$

Now, the solutions to the equation for G are trigonometric, and the zero condition at $x = 0$ means it is a sine function. The frequency is $\sqrt{k}$, so we need $\sqrt{k} = n\pi$, or $k = n^2\pi^2$ (for n a positive integer) to match the zero condition at $x = 1$. So, for each integer, we put $G_n(x) = \sin(n\pi x)$. Then we have $T''(t) + (5n\pi)^2 T(t) = 0$, so $T_n(t) = B_n \cos(5n\pi t) + B_n^* \sin(5n\pi t)$. This gives the product solution

$$u_n(x,t) = (B_n \cos(5n\pi t) + B_n^* \sin(5n\pi t))\sin(n\pi x).$$

To determine the constants B_n and B_n^*, we rely on the orthogonality relations of sine and cosine. The initial value is matched through the integrals

$$\begin{aligned} B_n =& 2\int_0^1 x\sin(n\pi x) - x^2 \sin(n\pi x)\,dx \\ =& 2\left[-x\cos(n\pi x) + x^2\cos(n\pi x)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x) - 2x\cos(n\pi x)\,dx \\ =& \frac{2}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^1 - \frac{4}{n^2\pi^2}\left[x\sin(n\pi x)\right]_0^1 + \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)\,dx \\ =& \frac{-4}{n^3\pi^3}\left[\cos(n\pi x)\right]_0^1 = \frac{4(1-(-1)^n)}{n^3\pi^3}. \end{aligned}$$

Thus, for $n = 2j$ it is zero, and for $n = 2j+1$ it is $\frac{8}{(2j+1)^3\pi^3}$. The initial derivative is matched through the integrals

$$B_n^* = \frac{2}{5n\pi}\int_0^1 \sin(\pi x)\sin(n\pi x)\,dx,$$

which yield 0 for all n except $n = 1$, for which we have $B_1^* = \frac{1}{5\pi}$ (recall the integral $\int_0^L \sin^2(\frac{n\pi}{L}x)\,dx = \frac{L}{2}$). Thus, our final solution is

$$u(x,t) = \frac{\sin(5\pi t)\sin(\pi x)}{5\pi} + \sum_{j=0}^{\infty} \frac{8\sin((10j+5)\pi t)\sin((2j+1)\pi x)}{(2j+1)^3\pi^3}.$$

(b) Again, we first examine $G(x)T(t)$. Here, we get the ODEs

$$\begin{cases} T'(t) = -4kT(t), \\ G''(x) + kG(x) = 0. \end{cases}$$

To match the zero condition at 0, we use sine functions for G, and to match the zero condition at π, we require that $k = n^2$. Thus, $G_n(x) = \sin(nx)$. Then we have $T_n(t) = B_n e^{-4n^2 t}$. The B_n are computed by the integrals

$$B_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\sin(nx) + \sin(3x)\sin(nx)\,dx.$$

These vanish except $B_1 = B_3 = 1$. Thus, our final solution is

$$u(x,t) = \sin(x)e^{-4t} + \sin(3x)e^{-36t}.$$

Theoretical

(a) Use the Fourier transform (in x) to solve the wave equation

$$u_{tt} = u_{xx}$$

on $\mathbb{R}$ with the initial conditions $u(x,0) = e^{-x^2}$, $u_t(x,0) = 0$. You may use that the Fourier transform of e^{-x^2} is $\frac{1}{\sqrt{2}}e^{-\omega^2/4}$, and leave your answer as an integral.

(b) Use the substitution $v = xu$ to solve the differential equation

$$xu_{tt} = xu_{xx} + 2u_x$$

on $0 < x < \pi$ with zero boundary and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

(a) Letting $\hat{u}$ denote the Fourier transform of u, we see that by taking the transform of the original PDE, we have

$$\hat{u}_{tt} + \omega^2 \hat{u} = 0.$$

Thus, $\hat{u}(\omega,t)$ is a trigonometric function. As $\hat{u}(\omega,0) = \frac{1}{\sqrt{2}}e^{-\omega^2/4}$ and $\hat{u}_t(\omega,0) = 0$, we have in fact the transformed solution $\hat{u}(\omega,t) = \frac{1}{\sqrt{2}}e^{-\omega^2/4}\cos(\omega t)$. Then

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\omega x} e^{-\omega^2/4} \cos(\omega t)\, d\omega.$$

(b) We compute $v_{tt} = xu_{tt}$ and $v_{xx} = xu_{xx} + 2u_x$. Thus, we have v solving the wave equation $v_{tt} = v_{xx}$, with zero boundary and initial conditions $v(x,0) = xf(x)$ and $v_t(x,0) = xg(x)$. Thus, v has a solution

$$v(x,t) = \sum_{n=1}^{\infty} (B_n \cos(nt) + B_n^* \sin(nt)) \sin(nx),$$

with $B_n = \frac{2}{\pi}\int_0^\pi xf(x)\sin(nx)\,dx$ and $B_n^* = \frac{2}{n\pi}\int_0^\pi xg(x)\sin(nx)\,dx$. The function u we see then has the solution

$$u(x,t) = \sum_{n=1}^{\infty} (nB_n \cos(nt) + nB_n^* \sin(nt)) \frac{\sin(nx)}{nx}.$$