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Math 8
Homework 6 Key

HW 6 Graded Problems

Pg 161 (9): Consider the following putative theorem:

Theorem: *There are irrational numbers a and b such that a^b is rational.*

Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

Proof: Either $\sqrt{2}^{\sqrt{2}}$ is rational or it's irrational.

Case 1. $\sqrt{2}^{\sqrt{2}}$ is rational. Let $a = b = \sqrt{2}$. Then a and b are irrational, and $a^b = \sqrt{2}^{\sqrt{2}}$, which we are assuming in this case is rational.

Case 2. $\sqrt{2}^{\sqrt{2}}$ is irrational. Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then a is irrational by assumption, and we know that b is also irrational. Also, $a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational.

This proof is correct, so the theorem is also correct. The main strategy used is a proof by cases, on the rationality or irrationality of $\sqrt{2}^{\sqrt{2}}$, which are exhaustive. In both cases, the proof produces an example of irrational numbers a and b such that a^b is rational, satisfying the theorem that such numbers exist. □

Pg 265 (11): Prove that for all $n \in \mathbb{N}$, $9 \mid (4^n + 6n - 1)$.

We proceed with a proof by induction. For $n = 0$, the expression $4^n + 6n - 1 = 1 + 0 - 1 = 0$, and $9 \mid 0$, establishing the base case. Suppose for some $n \in \mathbb{N}$ that $9 \mid (4^n + 6n - 1)$. Then there is an integer k such that $4^n + 6n - 1 = 9k$. We write $4^{n+1} + 6(n+1) - 1 = 4(4^n + 6n - 1) - 18n + 9 = 9(4k - 2n + 1)$. Thus, if $4^n + 6n - 1$ is divisible by 9, so is $4^{n+1} + 6(n+1) - 1$. This establishes for all $n \in \mathbb{N}$ that $9 \mid (4^n + 6n - 1)$. □

Alternatively, we could write $4^n = 9k - 6n + 1$, yielding $4^{n+1} + 6(n+1) - 1 = 4(9k - 6n + 1) + 6n + 5 = 36k - 18n + 9 = 9(4k - 2n + 1)$, to show the induction step works.

Pg 265 (13): Prove that for all integers a and b and all $n \in \mathbb{N}$, $(a+b) \mid (a^{2n+1} + b^{2n+1})$.

We proceed with a proof by induction. Let a and b be given. For $n = 0$, the expression $a^{2n+1} + b^{2n+1} = a + b$, which is clearly divisible by $a + b$, establishing the base case. Suppose for some $n \in \mathbb{N}$ that $(a+b) \mid (a^{2n+1} + b^{2n+1})$. Then there is an integer k such that $a^{2n+1} + b^{2n+1} = (a+b)k$. We write

$$a^{2(n+1)+1} + b^{2(n+1)+1} = a^2(a^{2n+1} + b^{2n+1}) - a^2 b^{2n+1} + b^2 b^{2n+1} = a^2(a+b)k - (a-b)(a+b)b^{2n+1} = (a+b)(a^2 k - (a-b)b^{2n+1}).$$

Thus, if $a^{2n+1} + b^{2n+1}$ is divisible by $a + b$, so too is $a^{2(n+1)+1} + b^{2(n+1)+1}$. This establishes for all $n \in \mathbb{N}$ that $(a+b) \mid (a^{2n+1} + b^{2n+1})$, and as a and b were arbitrary, this holds for all integers a and b. □

GS A: Prove there are infinitely many prime numbers of the form $3k+2$ for some integer k.

There are certainly primes of the form $3k+2$: $2 = 3(0)+2$ is one of them. Suppose $p_1, p_2, \ldots, p_n$ are primes of the form $3k+2$. Define $N = 3p_1p_2\cdots p_n - 1$. Since $N = 3(p_1p_2\cdots p_n - 1)+2$, it is of the form $3k+2$. Now, N leaves remainder -1 when divided by 3 or any p_i, so in particular it is not divisible by them.

If N is prime, it is not one of p_1 through p_n. If N is not prime, it is the product of primes. Since $3 \nmid N$, these primes are all of the form $3k+1$ or $3k+2$.

Since for all integers k_1 and k_2, $(3k_1+1)(3k_2+1) = 3(k_1k_2+k_1+k_2)+1$, the product of primes of the form $3k+1$ will also have that form. Since N is of the form $3k+2$, it must therefore have a factor q of the form $3k+2$. Since none of the p_i divide N, this q is not one of p_1 through p_n.

In both cases, we have found a prime not on our original list. Since n was arbitrary, we conclude no finite list contains all primes of the form $3k+2$, so there must be infinitely many of them. □

GS C Prove that for every integer $n \geq 2$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{7}{12}.$$

We proceed by induction. For $n=2$, $\frac{1}{2+1} + \frac{1}{2\cdot 2} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \geq \frac{7}{12}$, establishing the base case. Suppose for some $n \geq 2$ that $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{7}{12}$. Then we write $\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{2(n+1)} = -\frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$. Using the inductive hypothesis, this is greater than or equal to $-\frac{1}{n+1} + \frac{7}{12} + \frac{1}{2n+1} + \frac{1}{2n+2}$.

Now, $-\frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2} = \frac{-4n^2-3n-2+2n^2+2n+1+2n^2+3n+2}{(n+1)(2n+1)(2n+2)} = \frac{1}{(2n+1)(2n+2)} > 0$ for $n \geq 2$. Thus, $-\frac{1}{n+1} + \frac{7}{12} + \frac{1}{2n+1} + \frac{1}{2n+2} \geq \frac{7}{12}$. It follows that for all $n \geq 2$, $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{7}{12}$. □

Alternatively, we could argue that $\frac{1}{2n+1} > \frac{1}{2n+2}$ for $n \geq 2$, so $\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{2(n+1)} > \frac{1}{n+2} + \cdots + \frac{1}{2n} + \frac{1}{2n+2} + \frac{1}{2n+2}$, which is equal to $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, and then apply the inductive hypothesis.