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Math 8
Homework 9 Key

HW 9 Graded Problems

Pg 233 (2): Are the following functions?

(a) f defined by the provided graph.

(b) Let W be the set of English words, and A the set of letters in the (English) alphabet. Let $f = \{(w,a) \in W \times A : a \text{ is a letter in } w\}$ and $g = \{(w,a) \in W \times A : a \text{ is the first letter in } w\}$.

(c) John, Mary, Susan, and Fred go to dinner and sit at a round table. $P = \{\text{John, Mary, Susan, Fred}\}$, $R = \{(p,q) \in P \times P : p \text{ is sitting immediately to the right of } q\}$.

(a) No, because $d \notin \text{Dom}(f)$.

(b) f is not, because many words have more than one distinct letters. g is, because each word has a unique first letter.

[Note that a in the definition of f and g is a variable referring to any letter in the alphabet, not necessarily "a".]

(c) R is a function, because at a round table, each person has exactly one other to their right.

□

Pg 233 (8): Suppose A is a set. Show that i_A is the only relation on A which is both an equivalence relation and a function.

Since for all $a \in A$, $a = a$, and $a = b$ implies $b = a$, and $a = b$ and $b = c$ implies $a = c$, i_A is an equivalence relation. Since each element of A is only equivalent to itself, i_A is a function.

Suppose f is both an equivalence relation and a function. Then f is reflexive, so for all $x \in A$, $(x,x) \in f$. Since f is a function, this means $(x,y) \in f$ implies $y = x$ by uniqueness. So for all $x \in A$, $f(x) = x$, so $f = i_A$. Thus, i_A is the only equivalence relation that is also a function. □

Pg 233 (17):

(a) Suppose $g : A \to B$ and let $R = \{(x,y) \in A \times A : g(x) = g(y)\}$. Show that R is an equivalence relation on A.

(b) Suppose R is an equivalence relation on A and let $g : A \to A/R$ be defined by $g(x) = [x]_R$. Show that $R = \{(x,y) \in A \times A : g(x) = g(y)\}$.

(a) For all $x \in A$, $g(x) = g(x)$, so xRx, so R is reflexive. If $(x,y) \in R$, then $g(x) = g(y)$, so $g(y) = g(x)$, so $(y,x) \in R$, so R is symmetric. If $(x,y) \in R$ and $(y,z) \in R$, then $g(x) = g(y)$ and $g(y) = g(z)$, so $g(x) = g(z)$, so $(x,z) \in R$, so R is transitive. Thus, R is an equivalence relation.

(b) If xRy, then

$$z \in [x]_R \iff zRx \iff zRy \iff z \in [y]_R$$

(using transitivity for the middle equivalence), so $[x]_R = [y]_R$, so $g(x) = g(y)$, so $R \subseteq \{(x,y) \in A \times A : g(x) = g(y)\}$. If $g(x) = g(y)$, then $[x]_R = [y]_R$, so $x \in [x]_R \iff x \in [y]_R \iff xRy$, so $R \supseteq \{(x,y) \in A \times A : g(x) = g(y)\}$. So, $R = \{(x,y) \in A \times A : g(x) = g(y)\}$.

□

Pg 233 (19): Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 5\}$. Recall this is an equivalence relation.

(a) Show that there is a unique function $h : \mathbb{Z}/R \to \mathbb{Z}/R$ such that for every integer x, $h([x]_R) = [x^2]_R$.

(b) Show that there is no function $h : \mathbb{Z}/R \to \mathbb{Z}/R$ such that for every integer x, $h([x]_R) = [2^x]_R$.

(a) Let $f : \mathbb{Z} \to \mathbb{Z}/R$ be defined by $f(x) = [x^2]_R$. If xRy, then $x - y = 5k$, so $x^2 - y^2 = (x-y)(x+y) = 5k(x+y)$, so $x^2 R y^2$. So, if xRy, then $f(x) = [x^2]_R = [y^2]_R = f(y)$, so f is compatible with R. By problem 18, this means there is a unique h such that $h([x]_R) = [x^2]_R$.

[Note: we could define $h([x]_R) = [x^2]_R$; it was absolutely crucial to show that this actually defined a function, so we needed to check that if $[x]_R = [y]_R$, the definition gives the same value.]

(b) We know $[0]_R = [5]_R$. We can check $[2^0]_R = [1]_R$ and $[2^5]_R = [32]_R = [2]_R \neq [1]_R$. Thus, such an h would not be a function (it would need to take multiple values on the same equivalence class).

□

Pg 243 (6): Let $A = \mathscr{P}(\mathbb{R})$. Define $f : \mathbb{R} \to A$ by $f(x) = \left\{y \in \mathbb{R} : y^2 < x\right\}$.

(a) Find $f(2)$.

(b) Is f one-to-one? Is it onto?

(a) We see $f(2) = \left\{y \in \mathbb{R} : y^2 < 2\right\} = \left\{y \in \mathbb{R} : -\sqrt{2} < y < \sqrt{2}\right\} = (-\sqrt{2}, \sqrt{2})$.

(b) Since for all $y \in \mathbb{R}$, $y^2 \geq 0$, we see that $f(-1) = f(-2) = \emptyset$, so f is not injective.

If $z \in f(x)$, then $-z \in f(x)$, because $(-z)^2 = z^2 < x$ by hypothesis. Thus, the set $\{1\} \notin \mathrm{Ran}(f)$, because $1 \in \{1\}$ but $-1 \notin \{1\}$. Thus, f is not onto.

[Note: it's important to keep in mind that $f(x)$ is a *set*; while every $y \in \mathbb{R}$ is in $f(x)$ for some x, it is not the case that every $S \in \mathscr{P}(\mathbb{R})$ is an $f(x)$.]

□

Pg 243 (8): Suppose $f : A \to B$ and $g : B \to C$.

(a) Prove that if $g \circ f$ is onto, then g is onto.

(b) Prove that if $g \circ f$ is one-to-one, then f is one-to-one.

(a) Let $c \in C$ be arbitrary. If $g \circ f$ is onto, there is $a \in A$ such that $(g \circ f)(a) = c$. Since $(g \circ f)(a) = g(f(a))$, there is an element of B, namely $f(a)$, that g sends to c. Since c was arbitrary, this shows g is onto if $g \circ f$ is.

(b) Suppose for any $a_1, a_2 \in A$, $f(a_1) = f(a_2)$. Then $(g \circ f)(a_1) = g(f(a_1)) = g(f(a_2)) = (g \circ f)(a_2)$. If $g \circ f$ is one-to-one, this shows $a_1 = a_2$. Thus, if $g \circ f$ is one-to-one, f is.

□