Quiz 1

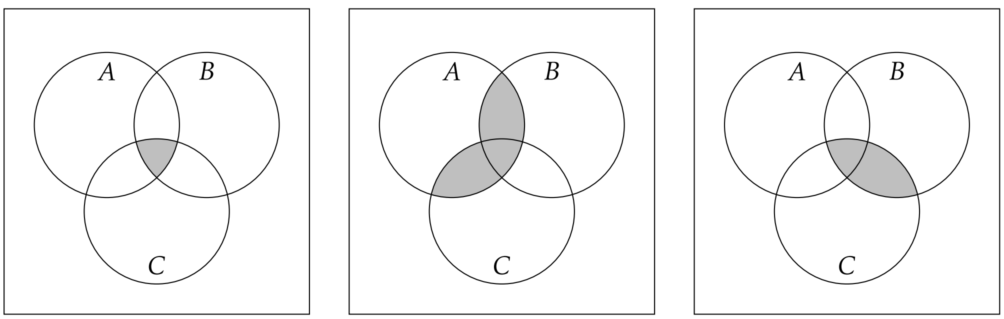
Assume A, B, and C are sets in some universe U, and define the set $X = (A \cap B) \cup (A \cap C)$. Further, let $P(x)$, $Q(x)$ and $R(x)$ denote the membership tests for A, B, and C, respectively (so, for example, $A = \{x \in U \mid P(x)\}$).

(a) Use Venn diagrams to determine if it must be the case that $A \cap B \cap C \subseteq X$ and $X \subseteq B \cap C$.

(b) With a sentence or two, explain why your answers to the above are correct.

(c) Prove your answers to the above are correct by working with the membership tests for each set (prove whether or not the membership test of the subset being true for some x implies the membership test of the superset must be true for that x, with no additional information).

(a) The Venn diagrams are these:



$A \cap B \cap C$ X $B \cap C$

From this, we can see that $A \cap B \cap C \subseteq X$, but $X \nsubseteq B \cap C$.

(b) Since everything in $A \cap B \cap C$ must be in X (in the diagram, the region for $A \cap B \cap C$ is contained in the region for X), it is a subset. To sketch a quick proof, being in $A \cap B \cap C$ means that certainly we're in $A \cap B$ (the condition is even stronger), which is one of the possibilities for being in X. So $A \cap B \cap C \subseteq X$.

However, it is possible to be in X and in just one of B or C, yet not both. Such an element would not be in $B \cap C$, so $X \nsubseteq B \cap C$ in general.

(c) The membership test for $A \cap B \cap C$ is $P(x) \wedge Q(x) \wedge R(x)$, and the membership test for X is $(P(x) \wedge Q(x)) \vee (P(x) \wedge R(x))$. We would like to show that

$$(P(x) \wedge Q(x) \wedge R(x)) \implies ((P(x) \wedge Q(x)) \vee (P(x) \wedge R(x)))$$

is a tautology. This is equivalent to showing

$$\neg(P(x) \wedge Q(x) \wedge R(x)) \vee ([P(x) \wedge Q(x)] \vee [P(x) \wedge R(x)])$$

by the definition of $\implies$, which is equivalent to

$$\neg(P(x) \wedge Q(x)) \vee \neg R(x) \vee (P(x) \wedge Q(x)) \vee (P(x) \wedge R(x))$$

by de Morgan's laws. This disjunction (chain of $\vee$s) includes the tautology $\neg(P(x) \wedge Q(x)) \vee (P(x) \wedge Q(x))$, and so is always true.

The membership test for $B \cap C$ is $Q(x) \wedge R(x)$. To show that $X \nsubseteq B \cap C$, it is enough to describe an x that meets the requirements for X, but not for $B \cap C$: that is, an x in the truth set of

$$([P(x) \wedge Q(x)] \vee [P(x) \wedge R(x)]) \wedge \neg(Q(x) \wedge R(x)).$$

If x satisfies $P(x) \wedge Q(x) \wedge \neg R(x)$, then $P(x) \wedge Q(x)$ is true and $Q(x) \wedge R(x)$ is false, so x satisfies the formula given. Thus, it is a counterexample to the inclusion $X \subseteq B \cap C$, and proves that it is false in general.

We can also argue by truth table:

$P(x)$	$Q(x)$	$R(x)$	$x \in A \cap B \cap C$	$x \in X$	$x \in B \cap C$
T	T	T	T	T	T
T	T	F	F	T	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	F	F	T
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F