Math 8
Worksheet 1

Logical Connectives

Mathematical proofs involve statements (things which are either true or false) which are combined using logical connectives (such as "and," "or," or "not") to form more complicated statements. An argument involves a collection of *premises* (statements which we assume are true), which lead to a conclusion through application of rules of logic. We call a conclusion *valid* if it is true whenever all of the premises are true. As shorthand for representing arguments, we often use letters to stand for simple statements (such as P for "n is a prime number" or G for "$x > 0$"), and symbols for logical connectives ($\wedge$ for "and," $\vee$ for "(inclusive) or," and $\neg$ for "not"). A formula in these symbols is *well-formed* if it can be interpreted as a statement (regardless of whether the statement is true).

1: Analyze the logical form of the following statements as a well-formed formula. Choose letters for simple statements and write down what each represents. Each simple statement should be a single true/false condition.

(a) An integer is either prime or composite.

(b) Seven is both negative and positive.

(c) $|x| \leq 1$ (use statements about x, such as $x < 1$).

(d) I did the reading, or I did not do the reading and the quiz was hard.

(e) Either $f(x) = 0$, or $f(x) \neq 0$ and $\frac{1}{f(x)}$ exists.

(a) We will put P for "the integer is prime" and C for "the integer is composite". Then the statement is $P \vee C$ (or possibly $(P \vee C) \wedge \neg(P \wedge C)$, although this is not necessary for the homework).

(b) We will put N for "seven is negative" and P for "seven is positive". Then the statement is $N \wedge P$—this is well-formed, even though it is invalid (seven is, of course, not negative).

(c) We will put L for $x < 1$, G for $x > -1$, P for $x = 1$, and N for $x = -1$. Then our statement is $(P \vee N) \vee (L \wedge G)$ (or $P \vee N \vee (L \wedge G)$, because of the associative property).

(d) We will put R for "I did the reading" and H for "the quiz was hard". Then we have $R \vee (\neg R \wedge H)$.

(e) We will put Z for $f(x) = 0$ and I for "$\frac{1}{f(x)}$ exists". Then we have $Z \vee (\neg Z \wedge I)$, the exact same logical structure as above.

2: Which of the following expressions are well-formed formulas? If they are, what is a statement with that logical form (it need not be mathematical)?

(a) $A \wedge (\neg B \vee C)$

(b) $A\neg \wedge B$

(c) $A \wedge \neg B$

(d) $A \vee \neg A$

(e) $\neg\neg A \wedge \neg A$

(f) $\vee A$

(a) This is well-formed. If A means "I am a TA", B means "I am busy", and C means "I am working", then this means "I am a TA and either I am not busy or I am working".

(b) This is not well-formed ($\neg$ needs to apply to a statement, but here it is applying to just the connector "and").

(c) This is well-formed. If A means "seven is positive" and B means "seven is negative", then this means "seven is positive and not negative".

(d) This is well-formed. If A means "x is an integer", then this means "x is either an integer or not an integer".

(e) This is well-formed. If A means "it is raining", then this means "it isn't not raining and it is not raining", which is always invalid (well-formed statements need not be true).

(f) This is not well-formed ($\vee$ needs to connect two statements, but only one is present).