Math 8
Worksheet 2

Equivalent Formulas

One of the simplest types of proof is to show that two statements are logically equivalent. In simple cases, this can be done explicitly, for example by constructing the relevant truth tables. Often, it is done by a chain of equivalences: working through intermediate statements which are all equivalent to each other to reach one of the statements from the other. It can sometimes be useful to write out expressions which explicitly encode all necessary conditions, and then reduce these to simpler equivalent expressions which are easier to work with.

1: Find simpler formulas equivalent to these formulas (and prove they are equivalent).
[The first three of these are taken from problem 11 on page 25 of Velleman]

(a) $\neg(\neg P \wedge \neg Q)$

(b) $(P \wedge Q) \vee (P \wedge \neg Q)$

(c) $\neg(P \wedge \neg Q) \vee (\neg P \wedge Q)$

(d) $(\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

(e) $(P \vee R) \wedge (\neg Q \vee R) \wedge (\neg P \vee R \vee Q)$

(f) $\neg\Big((\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)\Big) \wedge \Big((P \vee R) \wedge (\neg Q \vee R) \wedge (\neg P \vee R \vee Q)\Big)$

(a) Applying deMorgan's laws to this expression shows it is equivalent to $\neg\neg P \vee \neg\neg Q$, and applying the double negation law shows this is equivalent to $P \vee Q$.

(b) Applying one of the distributive laws to this expression shows it is equivalent to $P \wedge (Q \vee \neg Q)$, and the tautology law shows this is equivalent to P.

(c) Applying deMorgan's laws to this shows it is equivalent to $\neg P \vee \neg\neg Q \vee (\neg P \wedge Q)$, the double negation law shows it is equivalent to $\neg P \vee Q \vee (\neg P \wedge Q)$, and the absorption law shows it is equivalent to $\neg P \vee Q$. We have implicitly used an associative law to avoid parentheses with multiple $\vee$ connectives.

(d) A commutative law shows this is equivalent to $(\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge \neg Q)$, a distributive law gives $(\neg P \wedge (Q \vee \neg Q)) \vee \neg(P \wedge Q)$, a tautology law gives $\neg P \vee \neg(P \wedge Q)$, deMorgan's laws give $\neg(P \wedge (P \wedge Q))$, and absorption gives $\neg(P \wedge Q)$.

(e) A distributive law gives first $((P \wedge \neg Q) \vee R) \wedge (\neg P \vee R \vee Q)$, second $((P \wedge \neg Q) \wedge (\neg P \vee Q)) \vee R$, and third $((P \wedge \neg Q \wedge \neg P) \vee (P \wedge \neg Q \wedge Q)) \vee R$. Then tautology laws show this is equivalent to R because both possibilities on the left are always false.

(f) We have already shown this is equivalent to $\neg(\neg(P \wedge Q)) \wedge R$, and the double-negative law gives it is equivalent to $P \wedge Q \wedge R$.

2: Find formulas Φ and Ψ involving A, B, and C with the following truth table:

A	B	C	Φ	Ψ
T	T	T	T	F
T	T	F	F	T
T	F	T	F	T
T	F	F	T	T
F	T	T	F	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

For Φ, there are only three possible conditions under which Φ is true: all three of A, B, and C true, only A true, and only B true. So, the easiest way to create a formula Φ which works is $(A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C)$. This is clearly true in the three cases it should be, and in any case Φ should be false, at least one of the terms in each of the three statements joined by $\vee$ is false.

For Ψ, there are only two possible conditions under which Ψ is false: all three of A, B, and C are true, or all three are false. So, the easiest formula to create for Ψ is $\neg(A \wedge B \wedge C) \wedge \neg(\neg A \wedge \neg B \wedge \neg C)$. By deMorgan's laws this is equivalent to $(\neg A \vee \neg B \vee \neg C) \wedge (A \vee B \vee C)$. In any case where Ψ is true, both conditions hold, and when it is false one of them fails, so this is a correct formula.

Of course, these are not the only possible formulas, especially if we allow more connectives than $\wedge$, $\vee$, and $\neg$. Different possiblities for Φ are $(A \vee B) \wedge [\neg C \vee (A \wedge B \wedge C)]$ or $(A \vee B) \wedge [C \iff (A \wedge B)]$, and different possibilities for Ψ are $(A \vee B \vee C) \wedge \neg(A \wedge B \wedge C)$ or $\neg[(A \vee B \vee C) \implies (A \wedge B \wedge C)]$ or even $A + B + C$ (we need to show an associative law for + to use this). The two strategies shown (combining all the conditions for T using $\vee$, or the negations of all the conditions for F using $\wedge$) will always produce correct formulas, however, and are often useful if there are only a few conditions for T or F, respectively.