Conditionals

The last major pair of logical connectors is the conditional $P \implies Q$ and the biconditional $P \iff Q$ (our book uses single arrows $P \to Q$ and $P \leftrightarrow Q$ for these; this is just a difference in notation).

The first is read "if P, then Q", "P is a sufficient condition for Q", or "Q is a necessary condition for P", and is true when P is false (so the condition is not satisfied) or P and Q are both true (so the condition Q is true whenever the condition P is). This is logically equivalent to $\neg P \vee Q$ and $\neg Q \implies \neg P$.

The second is read "P if and only if Q" and sometimes written "P iff Q", and is true when P and Q have exactly the same truth value (both true or both false). It is logically equivalent to $(P \implies Q) \wedge (Q \implies P)$, $(P \implies Q) \wedge (\neg P \implies \neg Q)$, $(P \wedge Q) \vee (\neg P \wedge \neg Q)$, and $\neg(P + Q)$.

The *converse* of an implication $P \implies Q$ is $\neg P \implies \neg Q$, and the *contrapositive* is $\neg Q \implies \neg P$. A conditional is equivalent to its contrapositive, but not to its converse; a biconditional is equivalent to a conditional and its converse both being true.

1: Simplify the following expressions:

(a) $(P \iff Q) \wedge \neg Q$

(b) $(P \iff Q) \wedge (P + Q)$

(c) $(P \implies Q) \vee (P \implies R)$

(d) $(P \wedge Q) \implies (P \iff Q)$

(e) $((P \implies Q) \implies R) \wedge (P \implies (Q \implies R))$

(a) Since $P \iff Q$ is true precisely when P and Q share the same truth values, to satisfy this formula we need $\neg P$ to be true. So, an equivalent is the much simpler $\neg P \wedge \neg Q$, or $\neg(P \vee Q)$ by de Morgan's law.

(b) Since $P \iff Q$ requires to share truth values, and $P + Q$ requires them to differ, this expression is not satisfiable, and is the constant *false*.

(c) Here, we can write first $(\neg P \vee Q) \vee (\neg P \vee R)$. Then, the distributive law (and idempotence) gives this as $\neg P \vee Q \vee R$. If we like, we can apply the associative law and definition of the conditional to get this as $P \implies (Q \vee R)$ (a distributive law for $\implies$).

(d) Here, we can write first $(P \implies Q)$ as $(P \wedge Q) \vee (\neg P \wedge \neg Q)$. Then we can write $\neg(P \wedge Q) \vee (P \wedge Q) \vee (\neg P \wedge \neg Q)$. The first two terms form a tautology, so this is the constant *true*.

(e) Here, we write first $((\neg P \vee Q) \implies R) \wedge (P \implies (\neg Q \vee R))$. Then, we write $(\neg(\neg P \vee Q) \vee R) \wedge (\neg P \vee \neg Q \vee R)$. The distributive and de Morgan's laws give $(P \wedge \neg Q \wedge \neg(P \wedge Q)) \vee R$, and an absorption law gives $(P \wedge \neg Q) \vee R$. We can apply de Morgan's laws to get $\neg(\neg P \vee Q) \vee R$, or $(P \implies Q) \implies R$. The statement is not, however, equivalent to this, as this truth table shows.

P	Q	R	$P \implies Q$	$Q \implies R$	$(P \implies Q) \implies R$	$P \implies (Q \implies R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

2: Analyze the logical form of the following statements. If they are arguments, are they valid? (Problems (a) through (d) are exercise 1 on page 53 of Velleman.)

(a) If this gas either has an unpleasant smell or is not explosive, then it isn't hydrogen.

(b) Having both a fever and a headache is a sufficient condition for George to go to the doctor.

(c) Both having a fever and having a headache are sufficient conditions for George to go to the doctor.

(d) If $x \neq 2$, then a necessary condition for x to be prime is that x be odd.

(e) An integer n is either even or odd (but not both). If n is odd, then $n+1$ is even. If n is even or m is even, then nm is even. Thus, $n(n+1)$ is even.

(f) A sufficient condition for a function f to be continuous is for f to be constant. A necessary condition for f being differentiable is to be continuous. Thus, a discontinuous function f is not constant or not differentiable.

(g) A square s is in either set A or set B (but not both). If $s \in A$, then the triangle t is in set B. Thus, $A \neq \emptyset$.

(a) This has the logical form $(S \vee \neg E) \implies \neg H$, where S stands for "this gas has an unpleasant smell", E stands for "this gas is explosive", and H stands for "this gas is hydrogen."

(b) This has the logical form $(F \wedge H) \implies D$.

(c) This has the logical form $(F \implies D) \wedge (H \implies D)$ (which is equivalent to $(F \vee H) \implies D$.

(d) This has the logical form $(x \neq 2) \implies (P(x) \implies O(x))$.

(e) This is an argument. The premises are $E(n) + O(n)$, $O(n) \implies E(n+1)$, and $(E(n) \vee E(m)) \implies E(nm)$ and the conclusion is $E(n(n+1))$. Since the conclusion is true if $E(n)$ or $E(n+1)$ is true (by premise 3), and if $\neg E(n)$, then $O(n)$, thus $E(n+1)$, this argument is valid.

(f) Let $C(f)$ stand for "f is continuous", $K(f)$ for "f is constant", and $D(f)$ for "f is differentiable". Then our premises are $K(f) \implies C(f)$ and $D(f) \implies C(f)$, and our conclusion is $\neg C(f) \implies (\neg K(f) \vee \neg D(f))$. The conclusion is equivalent to $(\neg C(f) \implies \neg K(f)) \vee (\neg C(f) \implies \neg D(f))$, the disjuction of the contrapositives of our premises (which is true when they are). Thus, this argument is valid.

(g) Our premises are $(s \in A) + (s \in B)$ and $(s \in A) \implies (t \in B)$, and our conclusion is $\neg(A = \emptyset)$. However, if $s \in B$ and $t \in B$ and $A = \emptyset$, our premises are true but the conclusion false. Thus, this argument is invalid.