Quantifiers

The logical symbol $\forall x$ means "for all x", and means that the statement that follows is true for every value of x. The logical symbol $\exists x$ means "there exists x such that", and means that the statement that follows is true for at least one value of x. The symbol $\exists! x$ means "there exists a unique x such that". The statement $(\exists! x)P(x)$ is equivalent to $(\exists x)\Big(P(x) \wedge (\forall y)\big(P(y) \implies (y = x)\big)\Big)$.

We know that $\forall x \neg P(x)$ is equivalent to $\neg \exists x P(x)$ and $\exists x \neg P(x)$ is equivalent to $\neg \forall x P(x)$. This often allows us to simplify statements.

1: We say that a function f from $\mathbb{R}$ to $\mathbb{R}$ is *bounded* if there exists a positive real number M such that for all $x \in \mathbb{R}$, we have $|f(x)| \leq M$. If $\mathcal{F}$ is a set of functions, we say it is *uniformly bounded* if there exists a positive real number N such that for all $f \in \mathcal{F}$ and all $x \in \mathbb{R}$ we have $|f(x)| \leq N$.

Write positive logical formulas for both of these definitions and their negations. Prove or disprove the following statements:

(a) If $\mathcal{F}$ is a uniformly bounded set of functions, then for all $f \in \mathcal{F}$, f is bounded.

(b) If $\mathcal{F}$ is a set of functions, and for all $f \in \mathcal{F}$, f is bounded, then $\mathcal{F}$ is uniformly bounded.

(c) If there exists $f \in \mathcal{F}$ that is not bounded, then $\mathcal{F}$ is not uniformly bounded.

(d) If $\mathcal{F}$ is not uniformly bounded, then there exists $f \in \mathcal{F}$ that is not bounded.

2: If T is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, we say that T is *surjective* if

$$(\forall y \in \mathbb{R}^m)(\exists x \in \mathbb{R}^n)\big(y = T(x)\big).$$

We say that T is *injective* if

$$(\forall x \in \mathbb{R}^n)(\forall y \in \mathbb{R}^n)\Big(\big(T(x) = T(y)\big) \implies \big(x = y\big)\Big).$$

We say that T is *invertible* if

$$(\forall y \in \mathbb{R}^m)(\exists! x \in \mathbb{R}^n)\big(y = T(x)\big).$$

Prove that T is invertible iff it is both surjective and injective. [This will likely be very challenging.]