Math 8
Worksheet 5

Quantifiers

The logical symbol $\forall x$ means "for all x", and means that the statement that follows is true for every value of x. The logical symbol $\exists x$ means "there exists x such that", and means that the statement that follows is true for at least one value of x. The symbol $\exists! x$ means "there exists a unique x such that". The statement $(\exists! x)P(x)$ is equivalent to $(\exists x)\Big(P(x) \wedge (\forall y)\big(P(y) \implies (y = x)\big)\Big)$.

We know that $\forall x \neg P(x)$ is equivalent to $\neg \exists x P(x)$ and $\exists x \neg P(x)$ is equivalent to $\neg \forall x P(x)$. This often allows us to simplify statements.

1: We say that a function f from $\mathbb{R}$ to $\mathbb{R}$ is *bounded* if there exists a positive real number M such that for all $x \in \mathbb{R}$, we have $|f(x)| \leq M$. If $\mathcal{F}$ is a set of functions, we say it is *uniformly bounded* if there exists a positive real number N such that for all $f \in \mathcal{F}$ and all $x \in \mathbb{R}$ we have $|f(x)| \leq N$.

Write positive logical formulas for both of these definitions and their negations. Prove or disprove the following statements:

(a) If $\mathcal{F}$ is a uniformly bounded set of functions, then for all $f \in \mathcal{F}$, f is bounded.

(b) If $\mathcal{F}$ is a set of functions, and for all $f \in \mathcal{F}$, f is bounded, then $\mathcal{F}$ is uniformly bounded.

(c) If there exists $f \in \mathcal{F}$ that is not bounded, then $\mathcal{F}$ is not uniformly bounded.

(d) If $\mathcal{F}$ is not uniformly bounded, then there exists $f \in \mathcal{F}$ that is not bounded.

The function f is bounded means that

$$(\exists M > 0)(\forall x)(|f(x)| \leq M).$$

The set $\mathcal{F}$ is uniformly bounded means that

$$(\exists N > 0)(\forall f \in \mathcal{F})(\forall x)(|f(x)| \leq N).$$

The function f is not bounded means that

$$(\forall M > 0)(\exists x)(|f(x)| > M).$$

The set $\mathcal{F}$ is not uniformly bounded means that

$$(\forall N > 0)(\exists f \in \mathcal{F})(\exists x)(|f(x)| > N).$$

(a) We seek an $M > 0$ such that for all x, $|f(x)| \leq M$. The definition of uniformly bounded gives us some number $N > 0$. If $f \in \mathcal{F}$, then for all x we know that $|f(x)| \leq N$ by the definition of uniformly bounded. Thus, we can take $M = N$ to see that f is indeed bounded. This proves the statement given.

(b) If $\mathcal{F}$ is the set of all constant functions f_c, with $f_c(x) = c$ (this is a set indexed by $\mathbb{R}$), each one is bounded, but the set is not uniformly bounded. Specifically, if $N > 0$, then $|f_{N+1}(0)| = N + 1 > N$. This counterexample disproves the statement given.

(c) Given $N > 0$, we seek a $g \in \mathcal{F}$ and a $y \in \mathbb{R}$ such that $\left|g(y)\right| > N$. Since f is not bounded and $N > 0$, we know there exists $x \in \mathbb{R}$ such that $|f(x)| > N$ by the definition we gave of not bounded. Since $f \in \mathcal{F}$, taking $g = f$ and $y = x$ proves the statement. In fact, this is the contrapositive of the first statement.

(d) The set of constant functions f_c above is also a counterexample to this statement—the set is not uniformly bounded, but each function in it is bounded. In fact, this is the contrapositive of the second statement.

2: If T is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, we say that T is *surjective* if

$$(\forall y \in \mathbb{R}^m)(\exists x \in \mathbb{R}^n)\big(y = T(x)\big).$$

We say that T is *injective* if

$$(\forall x \in \mathbb{R}^n)(\forall y \in \mathbb{R}^n)\Big(\big(T(x) = T(y)\big) \implies \big(x = y\big)\Big).$$

We say that T is *invertible* if

$$(\forall y \in \mathbb{R}^m)(\exists! x \in \mathbb{R}^n)\big(y = T(x)\big).$$

Prove that T is invertible iff it is both surjective and injective. [This will likely be very challenging.]

We seek to show that if T is invertible, then it is both surjective and injective, and that if T is both surjective and injective, then it is invertible.

We will first show that if T is invertible, then it is surjective. Given any $y \in \mathbb{R}^m$, we must verify that there exists some $x \in \mathbb{R}^n$ such that $y = T(x)$. By the definition of invertibility, we are guaranteed a unique $x \in \mathbb{R}^n$ such that $y = T(x)$. This shows that at least one exists, so we're done.

Now we will show that if T is invertible, then it is injective. Suppose we are given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. If $T(x) \neq T(y)$, the conditional holds. If $T(x) = T(y)$, then let $w \in \mathbb{R}^m$ be that element (that is, let $w = T(x) = T(y)$). By the definition of invertibility, we are guaranteed a unique $z \in \mathbb{R}^n$ such that $w = T(z)$. Since $w = T(x)$ and $w = T(y)$ (by construction), uniqueness means $x = y = z$, so we're done.

Finally, we will show that if T is both surjective and injective, then it is invertible. Given $y \in \mathbb{R}^m$, we must verify there exists a unique $x \in \mathbb{R}^n$ such that $y = T(x)$. By surjectivity, there exists at least one x_0 such that $y = T(x_0)$ (this covers existence). Suppose that $y = T(x_1)$. Since $T(x_0) = T(x_1) = y$, by injectivity $x_0 = x_1$ (this covers uniqueness). Thus, T is invertible.