Math 8
Worksheet 6

More Set Theory

Set elements need not be "simple" objects like numbers. Sets of sets are often useful to consider. The most important is the *power set*: the power set of a set A is $\mathscr{P}(A) = \{S : S \subseteq A\}$. That is, $(\forall S \subset U)(S \in \mathscr{P}(A) \iff S \subseteq A)$.

We also talk about *indexed families* of sets: if for every element i of some *index set* I we have a set A_i, the indexed family is $\mathscr{A} = \{A_i : i \in I\}$. The family intersection $\bigcap \mathscr{A} = \bigcap_{i \in I} A_i = \{a : (\forall i \in I)(a \in A_i)\}$ and family union $\bigcup \mathscr{A} = \bigcup_{i \in I} A_i = \{a : (\exists i \in I)(a \in A_i)\}$ generalize the operations we have defined for pairs of sets.

1: Let $I = \{1, 2, 3\}$ and $A_i = \{x \in \mathbb{Z} : i \leq x \leq 2i\}$.

(a) What is $\mathscr{P}(A_i)$ for each i?

(b) Why does $\mathscr{P}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I}(\mathscr{P}(A_i))$? What is it?

(c) Why doesn't $\mathscr{P}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I}(\mathscr{P}(A_i))$? [Give an example of an element in one and not the other, and argue why that is the case.]

(a) We compute:

$$\mathscr{P}(A_1) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$
$$\mathscr{P}(A_2) = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\}\}$$
$$\mathscr{P}(A_3) = \{\emptyset, \{3\}, \{4\}, \{5\}, \{6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\}, \{3,4,5\}, \{3,4,6\}, \{3,5,6\}, \{4,5,6\}, \{3,4,5,6\}\}.$$

(b) We know that the power set of an intersection of two sets is the intersection of their power sets, so using the associative law we can show that the power set of an intersection of finitely many sets is the intersection of the power sets (the index set here is only three elements, so we only need to use this twice). We can also check it explicitly. Since $\bigcap_{i \in I} A_i = \emptyset$, $\mathscr{P}(\bigcap_{i \in I} A_i) = \mathscr{P}(\emptyset) = \{\emptyset\}$. We can also check that $\emptyset$ is the only common element in the three power sets we computed.

(c) The union of the power sets is the collection of all sets which are subsets of some A_i: that is,

$$\bigcup_{i \in I} \mathscr{P}(A_i) = \{S : (\exists i)(\forall s \in S)(s \in A_i)\}.$$

The power set of the union is the collection of all sets whose members are all in some A_i: that is,

$$\mathscr{P}\left(\bigcup_{i \in I} A_i\right) = \{S : (\forall s \in S)(\exists i)(s \in A_i)\}.$$

So, the set $\{1, 6\}$ is in the power set of the union, but not the union of the power sets, because $1 \in A_1$ and $1 \notin A_3$ and $6 \notin A_1$ and $6 \in A_3$.

2: Working in the universe $\mathbb{N}$, for all $n \in I = \{m : m \geq 2\}$ define the sets $M_n = \{m : m > n\} \cap \{p : (\exists q)(p = nq)\}$.

(a) Describe the members of the set M_n (in words).

(b) Prove that $\forall n \forall m (M_n \cap M_m \neq \emptyset)$.

(c) Prove that $\bigcap_{n \in I} M_n = \emptyset$.

(d) Prove that $\bigcup_{n \in I} M_n \neq \mathbb{N}$.

(e) What is the set $\mathbb{N} \setminus \bigcup_{n \in I} M_n$ (in words)? [Try writing out the first few terms.]

(a) The set M_n consists of all multiples of n other than itself.

(b) If $m = n$ the intersection is nonempty because for any set A, $A \cap A = A$. If $m \neq n$, the number $nm \in M_n$, as $m > 1$ so $nm > n$, and we can take $q = m$ in the definition. Similarly, $nm \in M_m$, as $n > 1$ so $nm > m$, and we can take $q = n$. So, $nm \in M_n \cap M_m$, so that set cannot be empty.

(c) In order for an integer m to be in $\bigcap_{n \in I} M_n$, it must be greater than n for all $n \in I$. However, $m < m + 1 \in I$, so this cannot be true. So, for all $m \in \mathbb{N}$, $m \notin \bigcap_{n \in I} M_n$; that is, the set is empty.

(d) To show this, we just need some $m \in \mathbb{N}$ that is not in $\bigcup_{n \in I} M_n$. One such is 2, because for all $n \in I$ we have $2 \leq n$, so it is not in the left set defining M_n.

(e) The first six terms of $\mathbb{N} \setminus \bigcup_{n \in I} M_n$ are 0, 1, 2, 3, 5, and 7. Zero, one, and two are members because they are smaller than all $n \in I$. The rest are larger than at least one $n \in I$, so must be in the set because they are not multiples of any numbers smaller than them (other than one): that is, because they are prime. [In fact, this is a technique for describing the set of prime numbers, known as the *Sieve of Eratosthenes*.]