Math 8
Worksheet 7

Proofs I

The most important thing to keep in mind when writing proofs is to make fully justified steps. To aid this, writing out the justifications (for example, writing "...since all of its hypotheses are met, we may apply Theorem 1, which tells us..." after checking the hypotheses of Theorem 1 and before stating its conclusion) can be helpful.

It is important never to assume that the conclusion of a theorem is true in its proof. To prove theorems with the logical form $P \implies Q$, we generally assume P is true during the proof and try to derive Q (if P is false, then $P \implies Q$ is true, so we don't need to check that case). Sometimes, it is helpful to instead prove the logically equivalent contrapositive $\neg Q \implies \neg P$. Often, theorems with the logical form $P \iff Q$ are easiest to prove as $(P \implies Q) \wedge (Q \implies P)$.

1: Prove the following:
Theorem 1: If a and b are non-negative real numbers and $a > b$, then $a^2 > b^2$.
Theorem 2: If a and b are non-negative real numbers and $a^2 > b^2$, then $a > b$.
Theorem 3: If a and b are non-negative real numbers, then $a > b$ if and only if $a^2 > b^2$.

For Theorem 1, if $b = 0$, then because $a > 0$ and $0^2 = 0$, $a^2 > 0^2$. Otherwise, $b > 0$, so $a > 0$. Multiplying both sides of an inequality by a positive number gives a new inequality in the same direction. Multiplying both sides of $a > b$ by a gives $a^2 > ab$, and by b gives $ab > b^2$. So, $a^2 > ab > b^2$, so in particular if $a > b$ then $a^2 > b^2$.

For Theorem 2, we examine the contrapositive: if $a \leq b$, then $a^2 \leq b^2$. If $a = b$, then $a^2 = b^2$. Otherwise, $a < b$, so applying Theorem 1 gives $a^2 < b^2$. In both cases $a^2 \leq b^2$. Thus, if $a^2 > b^2$ then $a > b$.

For Theorem 3, we must show that if $a > b$ then $a^2 > b^2$, and that if $a^2 > b^2$ then $a > b$. The first statement follows by Theorem 1, and the second by Theorem 2. Thus, $a > b$ if and only if $a^2 > b^2$.

2: Using the previous results, prove the Arithmetic-Geometric Mean Inequality:
Theorem 4: If a and b are non-negative real numbers, then $\frac{a+b}{2} \geq \sqrt{ab}$.

Since

$$\left(\frac{a+b}{2}\right)^2 = \frac{a^2+2ab+b^2}{4} = \frac{a^2-2ab+b^2+4ab}{4} = \left(\frac{a-b}{2}\right)^2 + ab,$$

and the squares of real numbers are non-negative, this shows $\left(\frac{a+b}{2}\right)^2 \geq ab$. For non-negative real numbers a and b, both $\frac{a+b}{2}$ and $\sqrt{ab}$ are non-negative real numbers, so by the contrapositive of Theorem 1, $\frac{a+b}{2} \geq \sqrt{ab}$.