Math 8
Worksheet 8

Proofs II

1: Prove the following:

(a) For integers n, n^2 is odd if and only if n is odd.

(b) If $\{x_k : 1 \leq k \leq n\}$ is a set of integers, and $\sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n$ is not divisible by n, then there exist i and j such that $x_i \neq x_j$.

2: What (if anything) is wrong with the following proofs?

Theorem: Given any positive integer n, $\frac{1}{2}n(n+1)$ is a positive integer.

(a) For $n = 5$, $\frac{1}{2}n(n+1) = \frac{1}{2}(5)(6) = 15$, which is a positive integer. So, if n is a positive integer, then $\frac{1}{2}n(n+1)$ is a positive integer.

(b) This theorem has the logical form $(\forall n > 0)(\frac{1}{2}n(n+1) \in \mathbb{Z}^+)$. This is equivalent to $\left\{\frac{1}{2}n(n+1) : n > 0\right\} \subseteq \mathbb{Z}^+$. The set on the left hand side is $\{1, 3, 6, 10, 15, \ldots\}$, whose elements are only positive integers. So, if n is a positive integer, then $\frac{1}{2}n(n+1)$ is a positive integer.

(c) Suppose $\frac{1}{2}n(n+1) = m \in \mathbb{Z}^+$. Then $2m \in \mathbb{Z}^+$, and $2m = n^2 + n$. Rearranging, $n^2 + n - 2m = 0$. Then the Quadratic Formula gives $n = \frac{-1 \pm \sqrt{1+8m}}{2}$. Since $\sqrt{1+8m} > 1$ for $m \in \mathbb{Z}^+$, there is always a solution with $n > 0$. So, if n is a positive integer, then $\frac{1}{2}n(n+1)$ is a positive integer.

(d) For a given positive integer n, $n+1$ is also positive. The product of positive numbers is positive, so $\frac{1}{2}n(n+1)$ is positive. If n is even, $\frac{1}{2}n$ is an integer. Otherwise, if n is odd, then $n+1$ is even and $\frac{1}{2}(n+1)$ is an integer. In both cases, we then multiply by an integer, and the product of integers is an integer. So, if n is a positive integer, then $\frac{1}{2}n(n+1)$ is a positive integer.

(e) Suppose $n = p/q$ is a rational number that isn't an integer. Then $\frac{1}{2}n(n+1) = \frac{p}{2q}\left(\frac{p+q}{q}\right) = \frac{p^2+pq}{2q^2}$, which is not always an integer. For example, when $p = 1$ and $q = 2$, it is $\frac{3}{8} \notin \mathbb{Z}$. So, if n is a positive integer, then $\frac{1}{2}n(n+1)$ is a positive integer.