Proofs II

1: Prove the following:

(a) For integers n, n^2 is odd if and only if n is odd.

(b) If $\{x_k : 1 \leq k \leq n\}$ is a set of integers, and $\sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n$ is not divisible by n, then there exist i and j such that $x_i \neq x_j$.

(a) Suppose n is odd. The product of odd numbers is odd, so n^2 is odd.

Conversely, suppose n is even (so not odd). The product of even numbers is even, so n^2 is even. Since we have shown that if n is odd then n^2 is odd, and that if n is not odd then n^2 is not odd, we can conclude that n^2 is odd if and only if n is odd.

(b) We will prove the contrapositive: if for all i and j, $x_i = x_j$, then $\sum_{k=1}^{n} x_k$ is divisible by n. Suppose for all i and j, $x_i = x_j$. In particular, for all k, $x_k = x_1$. Then $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} x_1 = nx_1$ (it is the sum of n copies of x_1). This is divisible by n. So, if $\sum_{k=1}^{n} x_k$ is not divisible by n, there must be some i and j so that $x_i \neq x_j$.

2: What (if anything) is wrong with the following proofs?

Theorem: Given any positive integer n, $\frac{1}{2}n(n+1)$ is a positive integer.

(a) For $n = 5$, $\frac{1}{2}n(n+1) = \frac{1}{2}(5)(6) = 15$, which is a positive integer. So, if n is a positive integer, then $\frac{1}{2}n(n+1)$ is a positive integer.

(b) This theorem has the logical form $(\forall n > 0)(\frac{1}{2}n(n+1) \in \mathbb{Z}^+)$. This is equivalent to $\left\{\frac{1}{2}n(n+1) : n > 0\right\} \subseteq \mathbb{Z}^+$. The set on the left hand side is $\{1, 3, 6, 10, 15, \ldots\}$, whose elements are only positive integers. So, if n is a positive integer, then $\frac{1}{2}n(n+1)$ is a positive integer.

(c) Suppose $\frac{1}{2}n(n+1) = m \in \mathbb{Z}^+$. Then $2m \in \mathbb{Z}^+$, and $2m = n^2 + n$. Rearranging, $n^2 + n - 2m = 0$. Then the Quadratic Formula gives $n = \frac{-1 \pm \sqrt{1+8m}}{2}$. Since $\sqrt{1+8m} > 1$ for $m \in \mathbb{Z}^+$, there is always a solution with $n > 0$. So, if n is a positive integer, then $\frac{1}{2}n(n+1)$ is a positive integer.

(d) For a given positive integer n, $n+1$ is also positive. The product of positive numbers is positive, so $\frac{1}{2}n(n+1)$ is positive. If n is even, $\frac{1}{2}n$ is an integer. Otherwise, if n is odd, then $n+1$ is even and $\frac{1}{2}(n+1)$ is an integer. In both cases, we then multiply by an integer, and the product of integers is an integer. So, if n is a positive integer, then $\frac{1}{2}n(n+1)$ is a positive integer.

(e) Suppose $n = p/q$ is a rational number that isn't an integer. Then $\frac{1}{2}n(n+1) = \frac{p}{2q}\left(\frac{p+q}{q}\right) = \frac{p^2+pq}{2q^2}$, which is not always an integer. For example, when $p = 1$ and $q = 2$, it is $\frac{3}{8} \notin \mathbb{Z}$. So, if n is a positive integer, then $\frac{1}{2}n(n+1)$ is a positive integer.

(a) Only $n = 5$ was checked, but the theorem requires the statement to be true for all positive integers.

(b) There's no argument why the set consists of only positive integers. All of the computed examples are, but there's no guarantee that further along in the sequence they will continue to be positive. This is just a slightly fancier version of the first "proof".

(c) There are two problems here. First, it isn't clear that the positive solution for n is an integer. We might be able to fix this by looking at the form of m (although this is likely to be circular, proving n is an integer by using the assumption it is an integer). More seriously, this proof assumes the conclusion and shows the hypotheses—this is the wrong direction. At best, it would establish the converse (if n is not a positive integer, $\frac{1}{2}n(n+1)$ is not either).

(d) This proof is correct.

(e) This is attempting to prove the converse. It also has the issue that the supposition is not the negation of the hypothesis (that n is not a positive integer), but only a subcase of that (we would also have to check irrational numbers and negative integers to establish the converse).