Math 8
Worksheet 9

Proofs III

1: Prove the following:

(a) Given an indexed family of sets $\{A_i : i \in I\}$ and $J \subseteq I$, $\bigcap_{i \in I} A_i \subseteq \bigcap_{j \in J} A_j$.

(b) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

(c) Given two families of sets $\mathcal{F}$ and $\mathcal{G}$, if $\bigcap \mathcal{F} = \bigcap \mathcal{G} \neq \emptyset$, then for all $F \in \mathcal{F}$, $G \in \mathcal{G}$, $F \cap G \neq \emptyset$.

2: What (if anything) is wrong with the following proofs? Are the theorems true or false?

(a) *Theorem*: Suppose $x \neq \pm 1$. Then if $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$, then $x \neq 2$.

Proof: Suppose $x = 2$. Then $\frac{x^2+1}{x^2-1} = \frac{4+1}{4-1} = \frac{5}{3}$, while $\frac{x+1}{x-1} = \frac{2+1}{2-1} = 3 \neq \frac{5}{3}$. This contradicts our supposition that $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$, so $x \neq 2$.

(b) *Theorem*: Suppose $x \neq \pm 1$ and $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$. Then $x \neq 2$.

Proof: Suppose $x = 2$. Then $\frac{x^2+1}{x^2-1} = \frac{4+1}{4-1} = \frac{5}{3}$, while $\frac{x+1}{x-1} = \frac{2+1}{2-1} = 3 \neq \frac{5}{3}$. This contradicts our supposition that $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$, so $x \neq 2$.

(c) *Theorem*: Suppose $x \neq \pm 1$. Then if $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$, then $x \neq 2$.

Proof: Suppose $x = 2$. Then $\frac{x^2+1}{x^2-1} = \frac{4+1}{4-1} = \frac{5}{3}$, while $\frac{x+1}{x-1} = \frac{2+1}{2-1} = 3 \neq \frac{5}{3}$. This shows the contrapositive of the theorem, so the theorem is true.

(d) *Theorem*: Suppose that $x \neq 0$ and $xy = 1 + x^2 y$. Then $y \neq 0$.

Proof: Since $xy = 1 + x^2 y$, we have $x(1-x)y = 1$. Since $x \neq 0$, $x(1-x) \neq 0$, so we can divide both sides by it to get $y = \frac{1}{x(1-x)} \neq 0$. So, $y \neq 0$.

(e) *Theorem*: Suppose that x and y are real numbers. If $x + y = 0$, then if $x > 0$ then $y < 0$.

Proof: Suppose $x + y = 0$ and $x \leq 0$. Then $x + y \leq 0 + y = y$, so $y \geq 0$. Thus, if $x > 0$, then $y < 0$.