Proofs III

1: Prove the following:

(a) Given an indexed family of sets $\{A_i : i \in I\}$ and $J \subseteq I$, $\bigcap_{i\in I} A_i \subseteq \bigcap_{j\in J} A_j$.

(b) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

(c) Given two families of sets $\mathcal{F}$ and $\mathcal{G}$, if $\bigcap \mathcal{F} = \bigcap \mathcal{G} \neq \emptyset$, then for all $F \in \mathcal{F}$, $G \in \mathcal{G}$, $F \cap G \neq \emptyset$.

(a) Suppose $x \in \bigcap_{i\in I} A_i$, and let $j \in J$ be arbitrary. Then, since $J \subseteq I$, $j \in I$. So, $x \in A_j$. Since j was arbitrary, $x \in \bigcap_{j\in J} A_j$.

(b) Suppose $S \in \mathscr{P}(A)$. If $x \in S$, then because $S \subseteq A$, $x \in A$. Then because $B \subseteq A$, $x \in B$. So, $S \subseteq B$, or $S \in \mathscr{P}(B)$. This works for any $S \in \mathscr{P}(A)$, so $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

(c) Let $F \in \mathcal{F}$ and $G \in \mathcal{G}$ be arbitrary. Since $\bigcap \mathcal{G}$ is nonempty, there is some $x \in \bigcap \mathcal{G}$, so in particular $x \in G$. Since $\bigcap \mathcal{G} = \bigcap \mathcal{F}$, $x \in \bigcap \mathcal{F}$, so in particular $x \in F$. This means that $x \in F \cap G$, so the set is nonempty. Since F and G were arbitrary, this works for all pairs of $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

2: What (if anything) is wrong with the following proofs? Are the theorems true or false?

(a) *Theorem*: Suppose $x \neq \pm 1$. Then if $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$, then $x \neq 2$.

Proof: Suppose $x = 2$. Then $\frac{x^2+1}{x^2-1} = \frac{4+1}{4-1} = \frac{5}{3}$, while $\frac{x+1}{x-1} = \frac{2+1}{2-1} = 3 \neq \frac{5}{3}$. This contradicts our supposition that $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$, so $x \neq 2$.

(b) *Theorem*: Suppose $x \neq \pm 1$ and $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$. Then $x \neq 2$.

Proof: Suppose $x = 2$. Then $\frac{x^2+1}{x^2-1} = \frac{4+1}{4-1} = \frac{5}{3}$, while $\frac{x+1}{x-1} = \frac{2+1}{2-1} = 3 \neq \frac{5}{3}$. This contradicts our supposition that $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$, so $x \neq 2$.

(c) *Theorem*: Suppose $x \neq \pm 1$. Then if $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$, then $x \neq 2$.

Proof: Suppose $x = 2$. Then $\frac{x^2+1}{x^2-1} = \frac{4+1}{4-1} = \frac{5}{3}$, while $\frac{x+1}{x-1} = \frac{2+1}{2-1} = 3 \neq \frac{5}{3}$. This shows the contrapositive of the theorem, so the theorem is true.

(d) *Theorem*: Suppose that $x \neq 0$ and $xy = 1 + x^2y$. Then $y \neq 0$.

Proof: Since $xy = 1 + x^2y$, we have $x(1-x)y = 1$. Since $x \neq 0$, $x(1-x) \neq 0$, so we can divide both sides by it to get $y = \frac{1}{x(1-x)} \neq 0$. So, $y \neq 0$.

(e) *Theorem*: Suppose that x and y are real numbers. If $x + y = 0$, then if $x > 0$ then $y < 0$.

Proof: Suppose $x + y = 0$ and $x \leq 0$. Then $x + y \leq 0 + y = y$, so $y \geq 0$. Thus, if $x > 0$, then $y < 0$.

(a) We never in fact supposed that $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$ *in our proof*, it is merely implied by the form of the theorem statement. Still, this is not a contradiction, so the proof is incorrect. The theorem is correct, though, as verified in part (c).

(b) Here, the theorem statement was not a conditional, but gave us as a hypothesis that $\frac{x^2+1}{x^2-1} = \frac{x+1}{x-1}$. So, the contradiction *is* of something we are assuming true at that point in the theorem, so this proof is correct.

(c) Here, we never claim a contradiction, just use the contrapositive. This proof is correct.

(d) We are not given that $x \neq 1$, so we cannot conclude $x(1-x) \neq 0$. Thus, the proof is incorrect. However, if $x = 1$, our equation becomes $y = 1 + y$, contradicting our supposition (from the theorem statement) that $xy = 1 + x^2y$, so in fact $x \neq 1$.

We can observe that, if $y = 0$, then $x(1-x)y = 0 \neq 1$. This is not quite the contrapositive—our conclusion is not a conditional. However, proving that if $xy = 1 + x^2y$ then $y \neq 0$ and knowing $xy = 1 + x^2y$ does prove $y \neq 0$, so this is a valid way of thinking about how to provide a correct proof.

(e) Supposing $x \leq 0$ will not help us with the desired theorem, only the converse. Instead, to see this is true suppose $x + y = 0$ and $y \geq 0$. Then $x + y \geq x + 0 = x$, so $x \leq 0$; this is the contrapositive, so we conclude the theorem statement is true.