Math 8
Worksheet 10

Proofs IV

1: Prove the following:

(a) (Velleman pg. 135 n. 21) Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\bigcup\mathcal{F} \nsubseteq \bigcup\mathcal{G}$, then there is some $A \in \mathcal{F}$ such that for all $B \in \mathcal{G}$, $A \nsubseteq B$.

(b) (Velleman pg. 134 n. 12) For all $x \in \mathbb{R}$, $x \neq 1$ is equivalent to there being $y \in \mathbb{R}$ such that $x + y = xy$.

(c) Suppose $|x-1| < 1$. Then $x > 0$.

(d) If $3 \mid n$, then the remainder of $n \div 6$ is either 0 or 3.

(a) Since $\bigcup\mathcal{F} \nsubseteq \bigcup\mathcal{G}$, there is some $x \in \bigcup\mathcal{F}$ such that $x \notin \bigcup\mathcal{G}$. Since $x \in \bigcup\mathcal{F}$, there is some $A \in \mathcal{F}$ such that $x \in A$. Let $B \in \mathcal{G}$ be arbitrary. Since $x \notin \bigcup\mathcal{G}$, $x \notin B$, so in particular $A \nsubseteq B$.

[An important note here is that we produced the set A before we chose B: it's important that A does not depend on B, so this order is important.]

(b) We start by showing if $x \neq 1$ there is $y \in \mathbb{R}$ such that $x + y = xy$. Let $y = \frac{x}{x-1}$ (this is well-defined because $x \neq 1$). Then

$$x + y = \frac{x(x-1)+x}{x-1} = \frac{x^2}{x-1} = x\left(\frac{x}{x-1}\right) = xy.$$

Now we show if $x = 1$, for all $y \in \mathbb{R}$, $x + y \neq xy$. In fact, $x + y = 1 + y$, while $xy = y$. Since $1 + y \neq y$, $x + y \neq xy$.

[Note that we simply presented y in the first direction, and then proved it worked, rather than deriving it in the proof.]

(c) First suppose $x - 1 \geq 0$. Then $x \geq 1 > 0$, so $x > 0$. Otherwise, suppose $x - 1 < 0$. Then $|x-1| = -(x-1) = 1 - x$. Then $1 - x < 1$; adding x and subtracting 1 from both sides gives $0 < x$. Since our cases for $x - 1$ were exhaustive, we conclude that $x > 0$.

(d) Since $3 \mid n$, we can write $n = 3k$ for some integer k. If k is even, we can write $k = 2j$ for some integer j, so $n = 6j$. Then the remainder of $n \div 6$ is zero. Otherwise, if k is odd, we can write $k = 2j + 1$ for some integer j, so $n = 6j + 3$. Then the remainder of $n \div 6$ is three. Our cases were exhaustive, so the remainder is either zero or three.

2: Give a counterexample to the following "theorems".

(a) If a, b, and c are real numbers, the polynomial $ax^2 + 2bx + c$ has two distinct real roots.

(b) Suppose $x \neq 0$ is given. Then for any real number y, there exists a real number z such that $z^2 < x + y$.

(c) If $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$, then there exists some $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that either $A \subseteq B$ or $B \subseteq A$.

(a) If $a = b = c = 0$, the polynomial $ax^2 + 2bx + c = 0$ for all x, so it has more than two roots. Alternately, if $a = 1$, $b = 0$, and $c = 1$, the polynomial $ax^2 + 2bx + c = x^2 + 1 \geq 1$ for all x, so it has no real roots. Alternately, if $a = 0$, $b = 1$, and $c = 0$, the polynomial $ax^2 + 2bx + c = 2x$ has a unique root at $x = 0$. Any of these three satisfy the requirements.

[Note: we can create families of counterexamples along these lines: if $b = 0$ and a and c are both positive, we have the second situation, and if $a = 0$, $b \neq 0$ we have the third. However, without choosing some particular value for all the constants a, b, and c, we don't have a counterexample. Sometimes this causes serious problems: if we just specified that $b = 0$ to get to the second situation, we could get polynomials like $x^2 - 1$, which do have two distinct real roots.]

(b) Here, we think of x as something we cannot choose, so our counterexample is some choice of y that can depend on x. For example, we can take $y = -x$; then $x + y = 0$. Since $z^2 \geq 0$ for all real z, there is no z as in the theorem.

[Note: nowhere do we use that $x \neq 0$. This isn't a problem; sometimes theorems have stronger requirements than are actually needed.]

(c) Here we need to give families $\mathcal{F}$ and $\mathcal{G}$ such that all choices of A and B do not satisfy the theorem. One way to do this is to put $\mathcal{F} = \{\{1,2\}\}$ and $\mathcal{G} = \{\{1,3\},\{2,3\}\}$. Then $\bigcup \mathcal{F} = \{1,2\}$, and $\bigcup \mathcal{G} = \{1,2,3\}$, so $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$, so the theorem should apply. However, the set $\{1,2\} \in \mathcal{F}$ is not a subset of either $\{1,3\}$ or $\{2,3\}$, and neither set in $\mathcal{G}$ can be a subset of it, because it doesn't include 3.

[Note: here, we have a particularly simple counterexample. We could also do something like setting $\mathcal{F} = \{\{i, i+1\} : i \in \mathbb{Z}\}$ and $\mathcal{G} = \{\{i, i+2\} : i \in \mathbb{Z}\}$. However, this would require a proof that no element of $\mathcal{F}$ is a subset of any element of $\mathcal{G}$ and vice versa; while in the smaller counterexample this could be checked by inspection. If possible, finite counterexamples are often easier to verify.]