Proofs V

1: Prove the following:

(a) If n^2 is odd, then $n^2 - 1$ is divisible by 8.

(b) If p and q are prime, $p \mid n$, and $q \mid n$, then $pq \mid n$.
[You may use without proof the fact that if $p \mid ab$ and p is prime, then $p \mid a$ or $p \mid b$. Is this true if p and q can be any integers?]

(c) For any real number x, $|x| \leq 1 + x^2$. [Consider cases based on $|x|$.]

(d) For any real number x, $|x| \leq \frac{1}{2}\left(1 + x^2\right)$. [As scratch work, consider manipulating the desired inequality to recognize a non-negative quantity.]

(a) Supposing n^2 is odd, we have previously shown n must be odd. Thus, $n = 2k + 1$ for some integer k, so $n^2 = 4k^2 + 4k + 1$. Then $n^2 - 1 = 4k(k+1)$. We have also shown that for all integers k, two divides $k(k+1)$. So, $k(k+1) = 2j$ for some integer j, so $n^2 - 1 = 4(2j) = 8j$. Thus, if n^2 is odd, $n^2 - 1$ is divisible by 8.

(b) Suppose p and q are prime numbers dividing n. Since $p \mid n$, we know that $n = pk$ for some integer k. Then $q \mid pk$, so $q \mid p$ or $q \mid k$. Since p is prime, $q \nmid p$, so $q \mid k$. Then $k = qj$ for some integer j, so $n = pqj$. Thus, pq divides n.

(c) If $|x| \leq 1$, since $x^2 \geq 0$ we have $|x| \leq 1 + x^2$. Otherwise, we have $|x| > 1$, so $x^2 = |x|^2 = |x| \cdot |x| > |x|$, so $|x| \leq 1 + x^2$. In both cases, $|x| \leq 1 + x^2$.

(d) We know that $0 \leq (1 - |x|)^2 = 1 - 2|x| + |x|^2$. Adding $2|x|$ to both sides gives $2|x| \leq 1 + |x|^2$, and $|x|^2 = x^2$, so $2|x| \leq 1 + x^2$. Dividing by two gives $|x| \leq \frac{1}{2}\left(1 + x^2\right)$.

2: A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if

$$(\forall x)(\forall \epsilon > 0)(\exists \delta > 0)\left(\left|y - x\right| < \delta \implies \left|f(y) - f(x)\right| < \epsilon\right).$$

Prove that if $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are both continuous, then the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = f(g(x))$ is continuous.

Fix $x \in \mathbb{R}$ and $\epsilon > 0$. Let $z = g(x)$. Since f is continuous, there exists $\gamma > 0$ such that if $|w - z| < \gamma$, then $|f(w) - f(z)| < \epsilon$. Since g is continuous, there exists $\delta > 0$ such that if $\left|y - x\right| < \delta$, then $\left|g(y) - g(x)\right| < \gamma$. So, if $\left|y - x\right| < \delta$, then $\left|g(y) - g(x)\right| = \left|g(y) - z\right| < \gamma$, so $\left|f(g(y)) - f(z)\right| = \left|h(y) - h(x)\right| < \epsilon$. This works for any x and ϵ, so h is continuous.