Math 8
Worksheet 12

Induction

The Principle of Mathematical Induction is a way to prove an ordered sequence of statements of the form $P(n)$ dependent on a variable $n \in \mathbb{N}$. An inductive proof has two steps: a *base case* and an *inductive step*.

The base case is the statement $P(n_0)$ for some particular n_0 (often but not always 0 or 1): we are establishing some true fact about a (hopefully simple) first case.

The inductive step is the implication $P(n) \implies P(n+1)$: we are establishing a tool that we can use to push our knowledge about early cases forward to later ones. Then we know for all $N \in \mathbb{N}$ that $P(N)$ is true, because we start with $P(n_0)$ and apply the inductive step to get

$$P(n_0) \implies P(n_0+1) \implies P(n_0+2) \implies \cdots \implies P(N-1) \implies P(N).$$

1: Prove the following:

(a) (Velleman pg. 235 n. 3) For all $n \in \mathbb{N}$, the sum $\sum_{k=0}^{n} k^3 = 0^3 + 1^3 + \cdots + n^3 = \left[\frac{1}{2}n(n+1)\right]^2$.

(b) Given a prime p, for all $n \in \mathbb{N}$, p divides one of the elements of $S_n = \{n, n+1, \ldots, n+(p-1)\}$.

(c) Prove that for all real numbers $x \neq 1$ and all natural numbers n, $\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$.

2: What is wrong with the following proof that everyone in the room has the same name?

Proof: We will induct on the number n of people in the room. Set $P(n)$ to mean "for any n people in the room, they all have the same name". We will use $n = 1$ as our base case; certainly $P(1)$ is true, as a person has a single name. Suppose $P(n)$ is true. Given n people $p_1, p_2, \ldots, p_n, p_{n+1}$, we apply $P(n)$ to the first n people to conclude $\text{name}(p_1) = \text{name}(p_2) = \cdots = \text{name}(p_n)$, and again to the last n people to conclude $\text{name}(p_2) = \cdots = \text{name}(p_n) = \text{name}(p_{n+1})$. Then by the transitive property of equality,

$$\text{name}(p_1) = \text{name}(p_2) = \cdots = \text{name}(p_n) = \text{name}(p_{n+1}),$$

proving $P(n+1)$. By the Principle of Mathematical Induction, then, no matter how many people are in the room, they all have the same name.