Math 8
Worksheet 12

Induction

The Principle of Mathematical Induction is a way to prove an ordered sequence of statements of the form $P(n)$ dependent on a variable $n \in \mathbb{N}$. An inductive proof has two steps: a *base case* and an *inductive step*.

The base case is the statement $P(n_0)$ for some particular n_0 (often but not always 0 or 1): we are establishing some true fact about a (hopefully simple) first case.

The inductive step is the implication $P(n) \implies P(n+1)$: we are establishing a tool that we can use to push our knowledge about early cases forward to later ones. Then we know for all $N \in \mathbb{N}$ that $P(N)$ is true, because we start with $P(n_0)$ and apply the inductive step to get

$$P(n_0) \implies P(n_0+1) \implies P(n_0+2) \implies \cdots \implies P(N-1) \implies P(N).$$

1: Prove the following:

(a) (Velleman pg. 235 n. 3) For all $n \in \mathbb{N}$, the sum $\sum_{k=0}^{n} k^3 = 0^3 + 1^3 + \cdots + n^3 = \left[\frac{1}{2}n(n+1)\right]^2$.

(b) Given a prime p, for all $n \in \mathbb{N}$, p divides one of the elements of $S_n = \{n, n+1, \ldots, n+(p-1)\}$.

(c) Prove that for all real numbers $x \neq 1$ and all natural numbers n, $\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$.

(a) Our base case here is $n = 0$, where both sides of the equation are 0. For our inductive step, we will assume that $\sum_{k=0}^{n} k^3 = \left[\frac{1}{2}n(n+1)\right]^2$. Now, $\sum_{k=0}^{n+1} k^3 = \sum_{k=0}^{n} k^3 + (n+1)^3$. Using our assumption, this is equal to $\left[\frac{1}{2}n(n+1)\right]^2 + (n+1)^3$. Factoring out $(n+1)^2$ and putting everything over a common denominator gives $(n+1)^2\left[\frac{n^2+4n+4}{4}\right] = \left[\frac{1}{2}(n+1)(n+2)\right]^2$. Thus, by the Principle of Mathematical Induction, $\sum_{k=0}^{n} k^3 = \left[\frac{1}{2}n(n+1)\right]^2$ for all $n \in \mathbb{N}$. (As an interesting fact, this shows that $\sum_{k=0}^{n} k^3 = \left[\sum_{k=0}^{n} k\right]^2$ for all $n \in \mathbb{N}$.)

(b) Our base case here is $n = 0$, for which $p \mid 0$. For our inductive step, we will assume p divides an element of $S_n = \{n, n+1, \ldots, n+(p-1)\}$. Consider the set $S_{n+1} = \{n+1, (n+1)+1, \ldots, [n+(p-1)]+1\} = \{n+1, n+2, \ldots, n+p\}$. If p divided an element of $\{n+1, n+2, \ldots, n+(p-1)\}$ in the inductive hypothesis, that element is in S_{n+1}, so we're done. Otherwise, $p \mid n$, so $n = pk$ for some k, so $n+p = p(k+1)$, so $p \mid n+p$. In both cases, p divides an element of S_{n+1}. Thus, for all $n \in \mathbb{N}$, p divides an element of S_{n+1}.

(c) Let $x \neq 1$ be arbitrary. For $n = 0$, we have $1 = \frac{1-x}{1-x} = 1$. Suppose $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$. Then

$$\sum_{k=0}^{n+1} x^k = \sum_{k=0}^{n} x^k + x^{n+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1} = \frac{1-x^{n+1}+x^{n+1}-x^{n+2}}{1-x} = \frac{1-x^{n+2}}{1-x}.$$

Thus, for all $x \neq 1$ and $n \in \mathbb{N}$, we have $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$.

2: What is wrong with the following proof that everyone in the room has the same name?

Proof: We will induct on the number n of people in the room. Set $P(n)$ to mean "for any n people in the room, they all have the same name". We will use $n = 1$ as our base case; certainly $P(1)$ is true, as a person has a single name. Suppose $P(n)$ is true. Given n people $p_1, p_2, \ldots, p_n, p_{n+1}$, we apply $P(n)$ to the first n people to conclude $\text{name}(p_1) = \text{name}(p_2) = \cdots = \text{name}(p_n)$, and again to the last n people to conclude $\text{name}(p_2) = \cdots = \text{name}(p_n) = \text{name}(p_{n+1})$. Then by the transitive property of equality,

$$\text{name}(p_1) = \text{name}(p_2) = \cdots = \text{name}(p_n) = \text{name}(p_{n+1}),$$

proving $P(n+1)$. By the Principle of Mathematical Induction, then, no matter how many people are in the room, they all have the same name.

The proof of the inductive step breaks down for $P(2)$: the names of everyone in the set $\{p_1\}$ are equal (there's only one), and in the set $\{p_2\}$ are equal. However, we can't use the transitive property of equality to show that $\text{name}(p_1) = \text{name}(p_2)$, because there's no link between the two sets ($\{p_1\} \cap \{p_2\} = \emptyset$, unlike for any $n > 2$). So the proof of the inductive step fails.