Math 8
Worksheet 13

Strong Induction

Strong Induction is a variant on induction, where as an inductive hypothesis, instead of assuming the previous case holds, we assume *all* previous cases hold. That is, we show the statement $[(\forall k < n)(P(k))] \implies P(n)$.

Alternatively, we can think of this as $\neg P(n) \implies [(\exists k < n)(\neg P(k))]$. That is, if case n is a counterexample to the theorem, there must be some previous counterexample.

Sometimes, we can do this without using specific properties of the cases k, at which point we don't need a base case. This is especially true if we use the counterexample formalism. Of course, it might not be true that we are able to show this with no properties, and may need one or more base cases (if, for example, we used an element of a strictly smaller subset in our proof, we would have to show the property directly for one element sets).

1: Prove the following:

(a) Suppose $a_{n+2} = 5a_{n+1} - 6a_n$. Show that if $a_1 = 1$ and $a_0 = 0$, then $a_n = 3^n - 2^n$ for all $n \in \mathbb{N}$.

(b) (Well-Ordering Principle) Every nonempty subset of the natural numbers has a least element.

(c) Suppose ϕ is a function of integers greater than one such that $\phi(nm) = \phi(n)\phi(m)$, and for all primes p, $\phi(p) < p$. Prove for all $n > 1$ that $\phi(n) < n$.

(d) Every positive integer $n > 1$ has the form $s^2 t$, where s is a positive integer and t is square-free (for all primes p, $p^2 \nmid t$).

(a) Here, we need the two given base cases (indeed, $0 = 3^0 - 2^0$, and $1 = 3^1 - 2^1$, so they are true). Suppose the formula holds for all $0 \leq k < n$. Then

$$a_n = 5a_{n-1} - 6a_{n-2} = 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2}) = (15-6)3^{n-2} - (10-6)2^{n-2} = 3^n - 2^n.$$

In fact, we only needed the formula to hold for the *two* previous cases, but we still phrase this using strong induction.

(b) Suppose $S \subset \mathbb{N}$ has no smallest element. Now, suppose $n \in S$ and for all $k < n$, $k \notin S$. Then n is the smallest element of S, a contradiction. Thus, $S = \emptyset$.

(c) Suppose $n > 1$ and for $1 < k < n$, $\phi(k) < k$. If n is prime, then $\phi(n) < n$ by definition. Otherwise, $n = ab$ for $1 < a < n$ and $1 < b < n$. Then $\phi(n) = \phi(a)\phi(b) < a\phi(b) < ab = n$ by inductive hypothesis. In both cases, $\phi(n) < n$.

(d) Suppose $n > 1$ and for all $1 < k < n$, k is the product of a square and a square-free integer. If n is prime, then n is square-free, so we're done. Otherwise, $n = ab$ for integers $1 < a < n$, $1 < b < n$.

Write $a = s_a^2 t_a$ and $b = s_b^2 t_b$ with t_a and t_b square-free, and let $d = \text{gcf}(t_a, t_b)$. Then we have $t_a = dT_a$ and $t_b = dT_b$, and $\text{gcf}(T_a, T_b) = 1$. Then $n = (s_a s_b d)^2 T_a T_b$. Let $s = s_a s_b d$ and $t = T_a T_b$. Then t is square-free, because T_a and T_b are square-free and they share no common factors. So $n = s^2 t$ for s an integer and t square-free.

2 (Recursion): Suppose there are 2^n students in a course, and the TA needs to alphabetize their homeworks to enter the grades on Gauchospace. Show that, if an "operation" is either dividing a stack of homeworks in two or combining two sorted stacks into one sorted stack, alphabetizing the homeworks can be done in $2^{n+1}-2$ operations.

Let $N(k)$ be the number of operations needed to sort a stack of 2^k homeworks. Since a stack with a single homework is a sorted stack, $N(0) = 0$. So sort a stack of 2^k homeworks, one can divide the stack in two with an operation, sort two stacks of 2^{k-1} homeworks with $N(k-1)$ operations, then combine the sorted stacks with another operation. So, $N(k) = 2 + 2N(k-1)$. Suppose inductively that $N(k-1) = 2^k - 2$. Then $N(k) = 2 + 2^{k+1} - 4 = 2^{k+1} - 2$. So, by the Principle of Induction, $N(n) = 2^{n+1} - 2$.