Relations

A *relation* between two sets X and Y is a subset of their product: $R \subseteq X \times Y$. That is, R consists of pairs of elements, one from X and one from Y. It can be helpful to write xRy to mean $(x,y) \in R$, to clarify our intuition that such elements are related.

The *domain* of a relation is $\mathrm{Dom}(R) = \{x \in X : (\exists y \in Y)(xRy)\}$, and the *range* is $\mathrm{Ran}(R) = \{y \in Y : (\exists x \in X)(xRy)\}$. The *inverse* of a relation is $R^{-1} = \{(y,x) \in Y \times X : xRy\}$, and the *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation $S \circ R = \{(x,z) \in X \times Z : (\exists y \in Y)(xRy \wedge ySz)\}$.

1: If $R \subset \mathbb{R} \times \mathbb{Z}$ is the relation defined by xRn iff $|x| \leq n$ and $S \subset \mathbb{R} \times \mathbb{Z}$ is the relation defined by xSn iff $x < n$, describe the following relations and find their domain and range.

(a) R^{-1}

(b) S^{-1}

(c) $R \circ R^{-1}$

(d) $R^{-1} \circ R$

(e) $R \circ S^{-1}$

(f) $S \circ R^{-1}$

(g) $S \circ S^{-1}$

(h) $S^{-1} \circ S$

2: Prove the following:

(a) If R is a relation, then $\text{Ran}(R^{-1}) = \text{Dom}(R)$.

(b) If $R \subseteq X \times Y$ and $S \subseteq X \times Y$, then $\{(x, y) \in X \times Y : xRy \wedge xSy\}$ is a relation from X to Y.

(c) In the situation above, $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.