Relations I

A *relation* between two sets X and Y is a subset of their product: $R \subseteq X \times Y$. That is, R consists of pairs of elements, one from X and one from Y. It can be helpful to write xRy to mean $(x, y) \in R$, to clarify our intuition that such elements are related.

The *domain* of a relation is $\text{Dom}(R) = \{x \in X : (\exists y \in Y)(xRy)\}$, and the *range* is $\text{Ran}(R) = \{y \in Y : (\exists x \in X)(xRy)\}$. The *inverse* of a relation is $R^{-1} = \{(y, x) \in Y \times X : xRy\}$, and the *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation $S \circ R = \{(x, z) \in X \times Z : (\exists y \in Y)(xRy \wedge ySz)\}$.

1: If $R \subset \mathbb{R} \times \mathbb{Z}$ is the relation defined by xRn iff $|x| \leq n$ and $S \subset \mathbb{R} \times \mathbb{Z}$ is the relation defined by xSn iff $x < n$, describe the following relations and find their domain and range.

(a) R^{-1} (c) $R \circ R^{-1}$ (e) $R \circ S^{-1}$ (g) $S \circ S^{-1}$

(b) S^{-1} (d) $R^{-1} \circ R$ (f) $S \circ R^{-1}$ (h) $S^{-1} \circ S$

(a) This is a relation from $\mathbb{Z}$ to $\mathbb{R}$, where $nR^{-1}x$ iff $n \geq |x|$. The domain is all integers n for which some real number has absolute value not greater than n; since n is a real number, this is true for all nonnegative n. The range is all real numbers which are less than some integer in absolute value, which is all real numbers. That is, $\text{Dom}(R^{-1}) = \mathbb{N}$ and $\text{Ran}(R^{-1}) = \mathbb{R}$.

(b) This is a relation from $\mathbb{Z}$ to $\mathbb{R}$ where $nS^{-1}x$ iff $n > x$. The domain is all integers (consider $x = n - 1$), and the range is all real numbers.

(c) This is a relation from $\mathbb{Z}$ to $\mathbb{Z}$, where n is related to m iff there is some real number x with $n \geq |x|$ and $|x| \leq m$: that is, some real number is less than both n and m in absolute value. This is true so long as both n and m are both nonnegative.

(d) This is a relation from $\mathbb{R}$ to $\mathbb{R}$, where x is related to y iff there is some integer n with $|x| \leq n$ and $n \geq \left|y\right|$: that is, with $|x|$ and $\left|y\right|$ both not greater than n. This is true for all real numbers.

(e) This is a relation from $\mathbb{Z}$ to $\mathbb{Z}$, where n is related to m iff there is some real number x with $n > x$ and $|x| \leq m$. Its domain is all integers, and its range is all non-negative integers.

(f) This is a relation from $\mathbb{Z}$ to $\mathbb{Z}$, where n is related to m iff there is some real number x with $n \geq |x|$ and $x < m$. Its domain is all non-negative integers, and its range is all integers.

(g) This is a relation from $\mathbb{Z}$ to $\mathbb{Z}$, where n is related to m iff there is some real number x with $n > x$ and $x < m$; that is, x is less than both n and m. The domain and range are all integers.

(h) This is a relation from $\mathbb{R}$ to $\mathbb{R}$, where x is related to y iff there is some integer n with $x < n$ and $n > y$. The domain and range are all real numbers.

2: Prove the following:

(a) If R is a relation, then $\operatorname{Ran}(R^{-1}) = \operatorname{Dom}(R)$.

(b) If $R \subseteq X \times Y$ and $S \subseteq X \times Y$, then $\{(x,y) \in X \times Y : xRy \wedge xSy\}$ is a relation from X to Y.

(c) In the situation above, $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

(a) We see that

$$x \in \operatorname{Ran}(R^{-1}) \iff \exists y\Big((x,y) \in R^{-1}\Big) \iff \exists y\,((y,x) \in R) \iff x \in \operatorname{Dom}(R).$$

Because the membership tests are equivalent, $\operatorname{Ran}(R^{-1}) = \operatorname{Dom}(R)$.

(b) The given set is a subset of $X \times Y$, so it is a relation. In particular, it is the relation $R \cap S \subseteq X \times Y$, as we can see by examining the membership tests.

(c) We see

$$(y,x) \in (R \cap S)^{-1} \iff (x,y) \in R \cap S \iff (x,y) \in R \wedge (x,y) \in S \iff (y,x) \in R^{-1} \wedge (y,x) \in S^{-1} \iff (y,x) \in R^{-1} \cap S^{-1}.$$

Since the membership tests are equivalent, the sets are equal.