Math 8
Worksheet 15

Relations II

A relation $R \subseteq X \times X$ is *reflexive* if for all $x \in X$, xRx. It is *symmetric* if for all $(x,y) \in R$, yRx. It is *transitive* if xRy and yRz implies xRz.

1: Prove or disprove the following statements. If they are false, can you find additional conditions to make them true?

(a) If R is transitive and $x_1, x_2, \ldots, x_n$ are such that $x_i R x_{i+1}$ for $i = 1, \ldots, n-1$ (that is, $x_1 R x_2, x_2 R x_3, \ldots, x_{n-1} R x_n$), then $x_1 R x_n$.

(b) (Velleman pg. 187 n. 13) If R_1 and R_2 are symmetric, then $R_1 \cup R_2$ is symmetric.

(c) (Velleman pg. 189 n. 22) If R is symmetric and transitive, then R is reflexive.

(a) We proceed with induction on n. If $n = 2$, we have the definition of transitivity. Suppose for any n elements the theorem holds. If we have $x_1, x_2, \ldots, x_n, x_{n+1}$ as in the statement of the theorem, by inductive hypothesis $x_1 R x_n$, and by assumption $x_n R x_{n+1}$. Then by transitivity, $x_1 R x_{n+1}$.

(b) Suppose $x(R_1 \cup R_2)y$. Then either xR_1y or xR_2y. Then because both relations are symmetric, either yR_1x or yR_2x. In both cases, $y(R_1 \cup R_2)x$. Since x and y were arbitrary, $R_1 \cup R_2$ is symmetric.

(c) This is false, because the definitions of symmetric and transitive relations quantify over the relation (for all $(x,y) \in R$), while the definition of reflexive quantifies over the base set (for all $x \in X$), so for example if $X = \{x\}$ and $R = \emptyset$, then R is symmetric and transitive, but not reflexive, because $(x,x) \notin R$. However, if we assume $\mathrm{Dom}(R) = X$, we may appeal to the following proof:

Let $x \in X$ be arbitrary. Then choose $y \in X$ such that xRy (possible because $\mathrm{Dom}(R) = X$). Then because R is symmetric, yRx. Since xRy and yRx, and R is transitive, xRx. Since x was arbitrary, this shows R is reflexive.

2: Are the following relations reflexive? symmetric? transitive?

(a) In any X, xRy iff $x \neq y$.

(b) In $\mathbb{Z}$, nRm iff $n \mid m$.

(c) In $\mathbb{R}^2$, $\mathbf{x}R\mathbf{y}$ iff $\det\left(\begin{bmatrix}\mathbf{x} & \mathbf{y}\end{bmatrix}\right) \geq 0$.

(d) For some X with a distinguished point $x \in X$, the relation on $\mathscr{P}(X)$ defined by ARB iff $x \in A \cap B$.

(e) For some X, the relation on $\mathscr{P}(X)$ defined by ARB iff $(A \cup B) \setminus (A \cap B) = \emptyset$.

(a) This is obviously not reflexive, and pretty clearly symmetric. It is not transitive in general: if $x = z$ and $x \neq y$, then $y \neq z$ (note: this is transitive if X has fewer than three elements).

(b) This is reflexive and transitive, but not symmetric.

(c) This is reflexive (if the columns of a matrix are linearly dependent, it has zero determinant) and not symmetric (if the columns of a matrix are swapped, the determinant switches sign). For transitivity, we can make an algebraic argument about the determinant being linear in each column. Alternatively, by Problem 1a in this sheet, if we come up with a chain of vectors, each R-related to the next, with the first not R-related to the last, R is not transitive. Consider $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}$. One can quickly verify that each is related to the next, but $\det\left(\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right) = -1$.

(d) This is symmetric ($A \cap B = B \cap A$) and transitive (if $x \in A \cap B$ and $x \in B \cap C$, in particular $x \in A$ and $x \in C$, so $x \in A \cap C$). It is not reflexive in general: if $A = X \setminus \{x\}$, then $A \cap A = A \not\ni x$.

(e) This is reflexive ($(A \cup A) \setminus (A \cap A) = A \setminus A = \emptyset$) and symmetric (union and intersection are symmetric). In fact, it is transitive: if $(A \cup B) \setminus (A \cap B) = \emptyset$, then $A \cup B \subseteq A \cap B$. It is always the case that $A \cap B \subseteq A \cup B$, so ARB iff $A \cap B = A \cup B$. This is true iff $A = B$ (work this out if it's not clear).