Math 8
Worksheet 16

Equivalence Relations

An *equivalence relation* is a reflexive, symmetric, and transitive relation. It defines a *partition* of the set X it is defined on: X is divided into subsets such that every element is in exactly one subset (the subsets are disjoint and cover all of X). We denote the subset containing $x \in X$ by $[x]$, and call it the *equivalence class* of X.

1: Prove the following statements:

(a) Suppose R is an equivalence relation on X. Define $S \subseteq \mathscr{P}(X) \times \mathscr{P}(X)$ by ASB iff for all $a \in A$, $b \in B$, aRb. Then S is an equivalence relation.

(b) Suppose R is an equivalence relation on X, and S is an equivalence relation on X/R. Prove there is a unique equivalence relation T on X such that xTy iff $[x]_R S [y]_R$, and that $R \subseteq T$ and $\bigcup [[x]_R]_S = [x]_T$.

[Essentially, we are showing that $(X/R)/S$ "looks like" X/T.]

(c) Suppose R is an equivalence relation on X. Then there is a unique equivalence relation T on A/R such that $[x]T[y]$ iff xRy.

[You may use the results of Velleman pg. 223 n. 13: if $A \subseteq B$ and R is an equivalence relation on A, then $S = R \cap (B \times B)$ is an equivalence relation on B with $[x]_S = [x]_R \cap B$. This is a special case of Velleman pg. 225 n. 23.]

2: Are the following equivalence relations?

(a) On M_n ($n \times n$ matrices), the relation $R = \{(A, B) : \{\text{eigenvalues of } A\} = \{\text{eigenvalues of } B\}\}$.

(b) On $\mathbb{N}^+$, the relation $R = \{(m, n) : m \text{ and } n \text{ have the same number of distinct prime factors}\}$.

(c) On $\mathbb{Z}$, for fixed odd prime p the relation $R_p = \{(m, n) : p \mid (m + n)\}$.

(d) On $\mathbb{R}^2$, the relation $R = \left\{ \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) : x_1 = y_1 \vee x_2 = y_2 \right\}$.