Equivalence Relations

An *equivalence relation* is a reflexive, symmetric, and transitive relation. It defines a *partition* of the set X it is defined on: X is divided into subsets such that every element is in exactly one subset (the subsets are disjoint and cover all of X). We denote the subset containing $x \in X$ by $[x]$, and call it the *equivalence class* of X.

1: Prove the following statements:

(a) Suppose R is an equivalence relation on X. Define $S \subseteq X/R \times X/R$ by ASB iff for all $a \in A$, $b \in B$, aRb. Then S is an equivalence relation.

(b) Suppose R is an equivalence relation on X, and S is an equivalence relation on X/R. Prove there is a unique equivalence relation T on X such that xTy iff $[x]_R S [y]_R$, and that $R \subseteq T$ and $\bigcup [[x]_R]_S = [x]_T$.

[Essentially, we are showing that $(X/R)/S$ "looks like" X/T.]

(c) Suppose R is an equivalence relation on X. Then there is a unique equivalence relation T on A/R such that $[x]T[y]$ iff xRy.

[You may use the results of Velleman pg. 223 n. 13: if $A \subseteq B$ and R is an equivalence relation on A, then $S = R \cap (B \times B)$ is an equivalence relation on B with $[x]_S = [x]_R \cap B$. This is a special case of Velleman pg. 225 n. 23.]

(a) We must check if S is reflexive, symmetric, and transitive. Suppose $A \in X/R$. Then for all $a \in A$, aRa because R is reflexive. So, ASA, so S is reflexive.

Suppose ASB, and let $a \in A$, $b \in B$ be arbitrary. By definition of S, aRb. Since R is symmetric, bRa. That is, because a and b were arbitrary, for all $b \in B$, $a \in A$, bRa. So, BSA, so S is symmetric.

Suppose ASB and BSC. Let $a \in A$, $b \in B$, and $c \in C$ be arbitrary. By definition of S, we know aRb and bRc. Since R is an equivalence relation, aRc. Since a and c were arbitrary, this shows ASC. Thus, S is transitive, hence an equivalence relation.

(b) For uniqueness, if T and T' satisfy the requirements of the theorem, then

$$xTy \iff [x]_R S [y]_R \iff xT'y,$$

so $T = T'$, as their membership tests are equivalent.

For existence, define $T = \{(x,y) \in X \times X : [x]_R S [y]_R\}$. This is a relation; we must show it is an equivalence relation. It is reflexive, as for $x \in X$, we know $[x]_R = [x]_R$, so $[x]_R S [x]_R$ because S is reflexive, so xTx by definition. It is symmetric, as if xTy, then $[x]_R S [y]_R$ by definition, so $[y]_R S [x]_R$ by symmetry of S, so yTx by definition. Finally, it is transitive, because if xTy and yTz, then $[x]_R S [y]_R$ and $[y]_R S [z]_R$ by definition, so because S is transitive $[x]_R S [z]_R$, so xTz by definition.

Suppose xRy. Then $[x]_R = [y]_R$, so $[x]_R S [y]_R$ by reflexivity of S, so xTy. Thus, xTy. Since x and y were arbitrary, this shows $R \subseteq T$.

Since $[x]_R \in \mathscr{P}(X)$, we know $[[x]_R]_S \subset \mathscr{P}(X)$; that is, it is a family of subsets of X, so the union makes sense, and $\bigcup [[x]_R]_S \in \mathscr{P}(X)$.

Now, suppose $y \in \bigcup [[x]_R]_S$. Then there is some $[z]_R \in [[x]_R]_S$ such that $y \in [z]_R$, which means there is some $z \in X$ such that $[z]_R S [x]_R$ and yRz. But then zTx and yTz (because $R \subseteq T$), so yTx, or $y \in [x]_T$. Thus, $\bigcup [[x]_R]_S \subseteq [x]_T$.

On the other hand, suppose $y \in [x]_T$. Then $[y]_R S [x]_R$ by definition, so $[y]_R \in [[x]_R]_S$. Since $y \in [y]_R$, this shows $y \in \bigcup [[x]_R]_S$. Thus, $[x]_T \subseteq \bigcup [[x]_R]_S$, so they are equal.

(c) For uniqueness, suppose T and T' are both equivalence relations on A/R as in the theorem. We have

$$[x]T[y] \iff xRy \iff [x]T'[y],$$

so $T = T'$ (their membership tests are logically equivalent). Thus, any such equivalence relation T is unique.

For existence, we have that $A/R \subset \mathscr{P}(X)$, so we can take S from the previous problem and create

$$T = S \cap ((A/R) \times (A/R)),$$

an equivalence relation on A/R.

Suppose $[x]T[y]$ for $[x],[y] \in A/R$. Then by definition of T, $[x]S[y]$. Then for all $x' \in [x]$ and $y' \in [y]$, $x'Ry'$. In particular, xRy.

Now suppose xRy, and let $x' \in [x]$, $y' \in [y]$ be arbitrary. Then $x'Rx$, xRy, and yRy', so by transitivity $x'Ry'$. So, $[x]S[y]$, and in particular $[x]T[y]$ because both were equivalence classes. Thus, $[x]T[y]$ iff xRy, so a T as in the theorem exists.

In fact, there is a simpler description of T: it is the identity relation on A/R. To see this, we note that $[x] = [y]$ iff for all x' and y' with $x'Rx$ and $y'Ry$, $x'Ry'$. In particular, xRy. Also, if xRy, then $[x] = [y]$ by transitivity.

2: Are the following equivalence relations?

(a) On M_n ($n \times n$ matrices), the relation $R = \{(A, B) : \{\text{eigenvalues of } A\} = \{\text{eigenvalues of } B\}\}$.

(b) On $\mathbb{N}^+$, the relation $R = \{(m, n) : m \text{ and } n \text{ have the same number of distinct prime factors}\}$.

(c) On $\mathbb{Z}$, for fixed odd prime p the relation $R_p = \{(m, n) : p \mid (m + n)\}$.

(d) On $\mathbb{R}^2$, the relation $R = \left\{\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) : x_1 = y_1 \vee x_2 = y_2\right\}$.

(a) Yes: it is reflexive, symmetric, and transitive by the reflexivity, symmetry, and transitivity of =.

(b) Yes: if $\nu : \mathbb{N}^+ \to \mathbb{N}$ is the function giving the number of distinct prime factors of an integer, this is mRn iff $\nu(m) = \nu(n)$. So, by reflexivity, symmetry, and transitivity of =, we have R is an equivalence relation.

(c) No: it is not reflexive; consider $m = p + 1$. Then $m + m = 2(p + 1)$, and p does not divide 2 or $p + 1$.

(d) No: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is related to both $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but those two are not related, so it is not transitive.