Math 8
Worksheet 17

Functions I

A *function* $f : X \to Y$ is a relation from X to Y where each $x \in X$ has exactly one element $y \in Y$ such that $(x, y) \in f$. Usually, this is denoted $y = f(x)$.

1: Prove the following statements:

(a) (Velleman pg 234 n 11c) Suppose $f : A \to B$ and S is a relation on B. Define

$$R = \{(x, y) \in A : f(x) S f(y)\}.$$

Prove that if S is transitive, R is as well.

(b) Suppose R is a relation from X to Y. Prove that the relation f from X to $\mathscr{P}(Y)$ defined as

$$\{(x, B) : (\forall y \in Y)(xRy \iff y \in B)\}$$

is a function.

(a) Suppose xRy and yRz. Then $f(x)Sf(y)$ and $f(y)Sf(z)$. Since S is transitive, this implies $f(x)Sf(z)$. By definition, this means xRz.

(b) We must show that for all x, if $(x, B) \in f$ and $(x, B') \in f$, then $B = B'$. If $(x, B) \in f$ and $(x, B') \in f$, then $y \in B \iff xRy \iff y \in B'$. Thus, $B = B'$.

2: Are the following functions?

(a) $\{(M,d) \in M_n \times \mathbb{R} : \det(M) = d\}$

(b) $\left\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^2\right\}$

(c) $\left\{(y,x) \in \mathbb{R} \times \mathbb{R} : y = x^2\right\}$

(d) $\left\{(y,x) \in \mathbb{R}^+ \times \mathbb{R} : y = x^2\right\}$

(e) $f : X \to X/R$ where $f(x) = [x]_R$ where R is an equivalence relation

(f) $f : X/R \to X$ where $f([x]_R) = x$ where R is an equivalence relation

(g) $f : X/R \to Y/S$ where R and S are equivalence relations, $g : X \to Y$ is a function, and $f([x]_R) = [g(x)]_S$

(a) Yes; a matrix has one determinant.

(b) Yes; a number has one square.

(c) No; $(1)^2 = (-1)^2$.

(d) Yes; a real number has one non-negative square root.

(e) Yes; the equivalence classes partition X, so each element is in exactly one equivalence class.

(f) No; in general, equivalence classes consist of more than one element.

(g) No; let $X = \{1,2\}$, $Y = \{a,b\}$, $g(1) = a$, $g(2) = b$, $R = X \times X$, $S = i_Y$. Then $f([1]_R) = [a]_S$ and $f([2]_R) = [b]_S$, but $[1]_R = [2]_R$ while $[a]_S \neq [b]_S$.