Functions II

A function $f : X \to Y$ is *one-to-one* if for all x_1 and x_2 in X, $f(x_1) = f(x_2)$ implies $x_1 = x_2$ (that is, each point of Y has at most one point in X mapped to it). It is *onto* if for all $y \in Y$ there exists $x \in X$ with $y = f(x)$ (that is, each point of Y has at least one point in X mapped to it). Of course, the above conditions are what is needed for a function to be *invertible* (the inverse relation f^{-1} is a function if each point of Y has exactly one point in X mapped to it).

1: Prove the following statements:

(a) Suppose $f, g : X \to Y$, and $h : Y \to Z$ are functions, with h one-to-one. Then $h \circ g = h \circ f$ implies $g = f$.

(b) Suppose $f, g : X \to Y$, and $h : W \to X$ are functions, with h onto. Then $g \circ h = f \circ h$ implies $g = f$.

(c) If $f : X \to X$ is also a transitive relation, then $f \circ f = f$.

(d) Suppose $f : X \to Z$, $g : Y \to Z$, and $h : X \to Y$, with h invertible. If $f = g \circ h$, then $f \circ h^{-1} = g$.

(e) Let X be a set, $\mathcal{F} = \{f \mid f : X \to X\}$, and $\mathcal{A} = \left\{f \in \mathcal{F} \mid f^{-1} \in \mathcal{F}\right\}$ ($\mathcal{A}$ is the collection of invertible functions). Define the relation R on $\mathcal{F}$ as $\left\{(f, g) \mid (\exists h \in \mathcal{A})(f = h^{-1} \circ g \circ h)\right\}$. Then R is an equivalence relation.

2: Are the following functions one-to-one? onto? If both, what is the inverse function?

(a) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = xy$.

(b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ (with $a, b \in \mathbb{R}$, $a \neq 0$).

(c) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$

(d) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ defined by $f(x) = \frac{1}{x}$