Functions II

A function $f : X \to Y$ is *one-to-one* if for all x_1 and x_2 in X, $f(x_1) = f(x_2)$ implies $x_1 = x_2$ (that is, each point of Y has at most one point in X mapped to it). It is *onto* if for all $y \in Y$ there exists $x \in X$ with $y = f(x)$ (that is, each point of Y has at least one point in X mapped to it). Of course, the above conditions are what is needed for a function to be *invertible* (the inverse relation f^{-1} is a function if each point of Y has exactly one point in X mapped to it).

1: Prove the following statements:

(a) Suppose $f, g : X \to Y$, and $h : Y \to Z$ are functions, with h one-to-one. Then $h \circ g = h \circ f$ implies $g = f$.

(b) Suppose $f, g : X \to Y$, and $h : W \to X$ are functions, with h onto. Then $g \circ h = f \circ h$ implies $g = f$.

(c) If $f : X \to X$ is also a transitive relation, then $f \circ f = f$.

(d) Suppose $f : X \to Z$, $g : Y \to Z$, and $h : X \to Y$, with h invertible. If $f = g \circ h$, then $f \circ h^{-1} = g$.

(e) Let X be a set, $\mathcal{F} = \{f \mid f : X \to X\}$, and $\mathcal{A} = \left\{f \in \mathcal{F} \mid f^{-1} \in \mathcal{F}\right\}$ ($\mathcal{A}$ is the collection of invertible functions). Define the relation R on $\mathcal{F}$ as $\left\{(f,g) \mid (\exists h \in \mathcal{A})(f = h^{-1} \circ g \circ h)\right\}$. Then R is an equivalence relation.

(a) Let $x \in X$ be arbitrary. Put $y_1 = g(x)$ and $y_2 = f(x)$. We are given $h(g(x)) = h(f(x))$. Thus, $h(y_1) = h(y_2)$. Since h is one-to-one, this implies $g(x) = y_1 = y_2 = f(x)$. Since x was arbitrary, this implies $g = f$.

[Technically speaking, we didn't need to rename the elements $g(x)$ and $f(x)$. However, this illustrates the application of the definition of one-to-one in terms of elements of the source and target spaces of h better.]

(b) Let $x \in X$ be arbitrary. Since h is onto, $x = h(w)$ for some $w \in W$. We are given $g(h(w)) = f(h(w))$. Thus, $g(x) = f(x)$. Since x was arbitrary, this implies $g = f$.

(c) Suppose f is a transitive relation and let $x \in X$ be arbitary. Put $y = f(x)$ and $z = f(y) = f(f(x))$. Since $(x,y) \in f$ and $(y,z) \in f$, and f is transitive, we have $(x,z) \in f$, or $z = f(x)$. Thus, $f(f(x)) = f(x)$. Since x was arbitrary, this implies $f \circ f = f$.

(d) Suppose $f = g \circ h$, and let $y \in Y$ be arbitrary. Because h is invertible, in particular it is onto, so $y = h(x)$ for some $x \in X$. Now, $f(h^{-1}(y)) = f(x)$ by definition of h^{-1}, $f(x) = g(h(x))$ by supposition, and $g(h(x)) = g(y)$ by definition of x. Since y was arbitrary, this implies $f \circ h^{-1} = g$.

(e) Since i_X is invertible, R is reflexive. By the previous problem (and a very similar exercise for composition on the left), R is symmetric. Suppose fRg and gRk. Then there exist invertible h_1 and h_2 such that $f = h_1^{-1} \circ g \circ h_1$ and $g = h_2^{-1} \circ k \circ h_2$. Thus, $f = h_1^{-1} \circ h_2^{-1} \circ k \circ h_2 \circ h_1$. Now, $h_1^{-1} \circ h_2^{-1} = (h_2 \circ h_1)^{-1}$ as relations, and is a function because it is the composition of functions. So, $h_2 \circ h_1 \in \mathcal{A}$ (that is, it is invertible), so fRk, so R is transitive. Thus, it is an equivalence relation.

2: Are the following functions one-to-one? onto? If both, what is the inverse function?

(a) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = xy$.

(b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ (with $a, b \in \mathbb{R}$, $a \neq 0$).

(c) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$

(d) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ defined by $f(x) = \frac{1}{x}$

(a) This is onto: for any $t \in \mathbb{R}$, $t = f\left(\begin{bmatrix} t \\ 1 \end{bmatrix}\right)$. It is not one-to-one: $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

(b) This is onto: for any $t \in \mathbb{R}$, $t = f\left(\frac{t-b}{a}\right)$. It is one-to-one: if $ax_1 + b = ax_2 + b$, subtracting b and dividing a from both sides yields $x_1 = x_2$. The inverse function is $f^{-1}(y) = \frac{y-b}{a}$.

(c) This is not onto: if $x \neq 0$, then $\frac{1}{x} \neq 0$. It is one-to-one: if $\frac{1}{x_1} = \frac{1}{x_2}$, because $0 \notin \mathrm{Ran}(f)$ we may apply the function f to both sides to see $x_1 = x_2$.

(d) This is onto: if $y \in \mathbb{R} \setminus \{0\}$, then $y = f(\frac{1}{y})$. It is one-to-one by the argument above. The inverse function is $f^{-1}(y) = \frac{1}{y}$.