Logical Propositions and Connectives

Mathematical proofs involve statements (things which are either true or false) which are combined using logical connectives (such as "and," "or," or "not") to form more complicated statements. As shorthand for representing arguments, we often use letters to stand for simple statements (such as P for "n is a prime number" or G for "$x > 0$"), and symbols for logical connectives ($\wedge$ for "and," $\vee$ for "(inclusive) or," and $\sim$ for "not"). A formula in these symbols is *well-formed* if it can be interpreted as a statement (regardless of whether the statement is true). A *proposition* is something which has a (definite, but not necessarily known) truth value.

1.1.1: Which of the following are propositions? Give the truth value for each proposition.

(a) What time is dinner?

(b) It is not the case that π is not a rational number.

(c) $x/2$ is a rational number.

(d) Either π is rational and 17 is prime, or $7 < 13$ and 81 is a perfect square.

(e) There are more than three false statements in the book, and this is one of them. [This question is taken from the book, and should be interpreted as a statement "in the book".]

(a) This is not a proposition.

(b) This is a proposition; if we put P for "π is a rational number", it has the form $\sim(\sim P)$, which is equivalent to P, and false (π is quite possibly the most famous irrational number).

(c) This is not a proposition as written: we cannot evaluate its truth without knowing what x is. We will explore the idea of variables more later.

(d) This is a (compound) proposition. Putting P for "π is a rational number" (again), Q for "17 is prime", R for "7 ¡ 13", and S for "81 is a perfect square", we have the logical form $(P \wedge Q) \vee (R \wedge S)$ (although there is some troublesome difficulty with the relationship between the English word "either" and the connective $\vee$). As indeed $7 < 13$ and $81 = 9^2$ is a perfect square, this is true despite the left factor $P \wedge Q$ being false.

(e) This is not a proposition, as while the left statement is verifiable, the right is not (if it is true, it means it is false, which means it is true...).

1.1.4/5: If P, Q, and R are true while S and K are false, which of the following are true? Consider this in the context of truth tables.

(a) $Q \wedge (R \wedge S)$

(b) $Q \vee (R \wedge S)$

(c) $(P \vee Q) \wedge (R \vee S)$

(d) $(\sim P \vee \sim Q) \vee (\sim R \vee \sim S)$

(e) $\sim P \vee \sim Q$

(f) $(\sim Q \vee S) \wedge (Q \vee S)$

(g) $(P \vee S) \wedge (P \vee K)$

(h) $K \wedge \sim (S \vee Q)$

(a) This is false, as the right factor is false, as S is false.

(b) This is true, as the left factor is true, as Q is true.

(c) This is true, as the left factor is true because P is true, and the right factor is true because R is true.

(d) This is true, as the right factor is true, as $\sim S$ is true, as S is false.

(e) This is false, as neither $\sim P$ nor $\sim Q$ is true, as P and Q are both true. This is equivalent to $\sim (P \wedge Q)$ by de Morgan's Laws, which is perhaps more obviously false.

(f) This is false, as the left factor is false, as $\sim Q$ is false and S is false.

(g) This is true, as the left factor and right factor are both true because P is true.

(h) This is false, as the left factor is false, as K is false.