Conditionals and Biconditionals

Most theorem statements rely on the conditional "if... then..." or "implies" and biconditional "if and only if..." or "is equivalent to...", denoted $\implies$ and $\iff$, respectively. The statement $P \implies Q$ is true if Q is true whenever P is (importantly, if P is known to be false, $P \implies Q$ is true regardless of whether Q is true or not), and $P \iff Q$ is true if P and Q share the same truth value (both true or both false).

1.2.5: Which of the following conditional sentences are true?

(a) If triangles have three sides, then squares have four sides?

(b) If hexagons have six sides, then the moon is made of cheese.

(c) If $7+6=14$, then $5+5=10$.

(d) The Nile River flows east only if 64 is a perfect square.

(a) This is true: triangles have three sides, so we must check that squares have four, but this is true.

(b) This is false: hexagons have six sides, so we must check the composition of the moon, which is stone and not cheese.

(c) This is true: $7+6 \neq 14$, so we do not even require $5+5=10$.

(d) This is false: $64 = 8^2$, and yet the Nile primarily flows north.

1.2.6: Which of the following are true?

(a) Triangles have three sides iff squares have four sides.

(b) $7+5=12$ if and only if $1+1=2$.

(c) $5+6=6+5$ iff $7+1=10$.

(d) A parallelogram has three sides iff 27 is prime.

(a) This is true, as triangles have three sides, and squares four.

(b) This is true, as $7+5=12$ and $1+1=2$.

(c) This is false, as $5+6=11=6+5$, but $7+1=8\neq 10$.

(d) This is true, as a parallelogram has four sides and $27=3^3$.

1.2.13: Give, if possible, an example of a true conditional sentence for which

(a) the converse is true.

(b) the converse is false.

(c) the contrapositive is false.

(d) the contrapositive is true.

(a) Here any one implication of a biconditional would do; for a mathematical example, "the linear transformation T is invertible if $\det(T) \neq 0$" works.

(b) An example here is "if f is a polynomial function, f is differentiable", as there are a great many differentiable functions which are not polynomial.

(c) This cannot be done, as the contrapositive of a conditional is equivalent to the original statement.

(d) Any conditional sentence will do here; one choice is (given some fixed number x) "if $x > 0$, then $x \neq 0$". The contrapositive, here "if $x = 0$, then $x \not> 0$" (or the more naturally phrased "if $x = 0$, then $x \leq 0$"), can be directly verified.

1.2.16: Determine whether each of the following is a tautology, a contradiction, or neither.

(a) $[(P \implies Q) \implies P] \implies P$

(b) $P \equiv P \wedge (P \vee Q)$

(c) $P \implies Q \iff P \wedge \sim Q$

(d) $P \implies [P \implies (P \implies Q)]$

(a) If P is false, $P \implies Q$ is true, so $(P \implies Q) \implies P$ is false, so the top implication is true. If P is true and Q is false, $P \implies Q$ is false, so $(P \implies Q) \implies P$ is true, so the top implication is true. If P is true and Q is true, every implication is true. Thus, this is a tautology.

(b) It is always true that $P \equiv P$, so this is equivalent to $P \vee Q$, which may be either true or false, so this is neither a tautology nor a contradiction.

(c) By de Morgan's Laws, $P \wedge \sim Q$ is equivalent to $\sim(\sim P \vee \sim\sim Q)$, which is equivalent to $\sim(\sim P \vee Q)$, which we know is equivalent to $\sim(P \implies Q)$. Thus, this is equivalent to $(P \implies Q) \iff \sim(P \implies Q)$, which is clearly a contradiction.

(d) If P is false, the top implication is true. If P is true and Q is false, $P \implies Q$ is false, so $P \implies (P \implies Q)$ is false, so the top implication is false. Thus, this is neither a tautology nor a contradiction.