Equivalent Forms

Here we are interested in determining if two formulas are necessarily equivalent (based on their logical structure and references to the same simpler propositions).

1.1.9: Suppose P, Q, R, and S are propositional forms, P is equivalent to Q, and R is equivalent to S. For each pair of forms, determine whether they are necessarily equivalent. If they are, explain why.

(a) P and R

(b) P and $\sim\sim Q$

(c) $P \wedge S$ and $Q \wedge R$

(d) $P \vee S$ and $Q \vee R$

(e) $\sim (P \wedge S)$ and $\sim Q \vee \sim R$

(f) $P \wedge Q$ and $S \wedge R$

(a) These need not be equivalent, as P and R are not necessarily related.

(b) These are equivalent, as $\sim\sim Q$ is equivalent to $\sim\sim P$, which is equivalent to P (as a double negation).

(c) These are equivalent, as $P \wedge S$ is equivalent to $P \wedge R$, which is equivalent to $Q \wedge R$.

(d) Similarly, these are equivalent, as $P \vee S$ is equivalent to $Q \vee S$, which is equivalent to $Q \vee R$.

(e) These are equivalent, as $\sim (P \wedge S)$ is equivalent to $\sim P \vee \sim S$ by a de Morgan's law, and this is equivalent to $\sim Q \vee \sim R$ in the same way as the previous parts.

(f) These need not be equivalent, as $P \wedge Q$ is equivalent to $P \wedge P$, which is equivalent to P, and $S \wedge R$ is equivalent to R similarly, and as observed above, P and R are not necessarily related.

1.2.8: Prove the following parts of Theorem 1.2.2 by showing the following pairs of statements are equivalent for propositions P and Q.

(a) $P \implies Q$ and $\sim P \vee Q$

(b) $P \iff Q$ and $(P \implies Q) \wedge (Q \implies P)$

(c) $\sim (P \implies Q)$ and $P \wedge \sim Q$

(d) $\sim (P \wedge Q)$ and $P \implies \sim Q$

(e) $\sim (P \wedge Q)$ and $Q \implies \sim P$

(f) $P \implies (Q \implies R)$ and $(P \wedge Q) \implies R$

(a) Here we appeal directly to the truth table:

P	Q	$P \implies Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	F	T	T

(b) Again, we construct a truth table, the way we defined these objects:

P	Q	$P \iff Q$	$P \implies Q$	$Q \implies P$	$(P \implies Q) \wedge (Q \implies P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

(c) Here, we can use the first part, getting that $\sim (P \implies Q)$ is equivalent to $\sim (\sim P \vee Q)$; then we can use a de Morgan's law to get $\sim\sim P \wedge \sim Q$, which is equivalent to $P \wedge \sim Q$.

(d) Here, we can use a de Morgan's law to get $\sim P \vee \sim Q$, then use the first part to get $P \implies \sim Q$.

(e) Here, we can use the previous part to get $P \implies \sim Q$, then take the contrapositive (which is equivalent) to get $\sim\sim Q \implies \sim P$, which is equivalent to $Q \implies \sim P$.

(f) Here, we can repeatedly use the first part to get $\sim P \vee (\sim Q \vee R)$, which is equivalent to $(\sim P \vee \sim Q) \vee R$. Then a de Morgan's law gives $\sim (P \wedge Q) \vee R$, which (again by the first part, which is really one of the most useful of these "algebraic" manipulations) gives $(P \wedge Q) \implies R$.