Quantifiers

When referencing variables, we bind them with quantifiers. There are two basic quantifiers, the universal quantifier, usually read "for all" and written ∀, and the existential quantifier, usually read "there exists" and written ∃.

1.3.10: Which of the following are true in the universe of all real numbers?

(a) $(\forall x)(\exists y)(x + y = 0)$

(b) $(\exists x)(\forall y)(x + y = 0)$

(c) $(\exists x)(\exists y)(x^2 + y^2 = -1)$

(d) $(\forall x)[x > 0 \implies (\exists y)(y < 0 \land xy > 0)]$

(e) $(\forall y)(\exists x)(\forall z)(xy = xz)$

(f) $(\exists x)(\forall y)(x \le y)$

(a) This is true: given an $x \in \mathbb{R}$, choosing $y = -x$ ensures $x + y = x + (-x) = 0$. In fact, $-x$ is the *unique* choice for y making the statement true.

(b) This, on the other hand, is false: such an x would have to be an additive inverse for *all* real numbers, but the inverse depends on the number. For any choice of x, the choice $y = 1 - x$ has $x + y = x + (1 - x) = 1 \neq 0$, disproving the statement.

(c) This is false as well: the statement $(\forall x)(x^2 \ge 0)$ is true, and $(\forall x)(\forall y)(x \ge 0 \land y \ge 0 \implies x + y \ge 0)$ is true; combining these, we have $(\forall x)(\forall y)(x^2 + y^2 \ge 0)$, and $-1 < 0$.

(d) This is false: if $x > 0$ and $y < 0$, then $xy < 0$.

(e) This is true: given any y, setting $x = 0$ makes $xy = xz = 0$ true for all choices of z. In fact, the stronger statement $(\exists x)(\forall y)(\forall z)(xy = xz)$ is true, because our choice of x did not depend on y.

(f) This is false, as for any choice of x, the number $y = x - 1 < x$.

1.3.13: Which of the following are denials of $(\exists! x)P(x)$?

(a) $(\forall x)P(x) \vee (\forall x) \sim P(x)$

(b) $(\forall x) \sim P(x) \vee (\exists y)(\exists z)(y \neq z \wedge P(y) \wedge P(z))$

(c) $(\forall x)[P(x) \implies (\exists y)(P(y) \wedge x \neq y)]$

(d) $\sim (\forall x)(\forall y)[(P(x) \wedge P(y)) \implies x = y]$

(a) This is not a *denial* of the statement (although if true in a universe of more than one element, it does make it false). Suppose $U = \{a, b, c\}$ and the truth set of P is $\{a, b\}$. Then $(\forall x)P(x)$ is false, as is $(\forall x) \sim P(x)$. Thus, in U, the given statements are not equivalent.

(b) This is a denial of the statement. If $(\forall x) \sim P(x)$ is true, $(\exists! x)P(x)$ is false. Otherwise, $(\exists y)P(y)$ is true; for $(\exists! x)P(x)$ to be false we then need some $z \neq y$ such that $P(z)$; this is exactly the second clause of the given statement.

(c) This is a simpler denial: if $(\forall x) \sim P(x)$, the implication is true because the hypothesis is false. If $(\exists x)P(x)$, the implication requires $y \neq x$ such that $P(y)$, so x is not unique.

(d) This is not a denial: if $(\forall x) \sim P(x)$, the statement $(\forall x)(\forall y)[(P(x) \wedge P(y)) \implies x = y]$ is true, so the given statement is false; however, $(\exists! x)P(x)$ is also false.

1.3.3: Translate these definitions from the *Appendix* into quantified sentences.

(a) The natural number *a divides* the natural number b.

(b) The natural number n is *prime*.

(c) The natural number n is *composite*.

(a) Here we write (in the universe $\mathbb{N}$)

$$a \mid b \iff (\exists q)(b = qa).$$

(b) Here, we have

$$\text{Prime}(n) \iff [n > 1 \wedge (\forall m)(m \mid n \implies (m = 1 \vee m = n))].$$

(c) Here, we could write

$$\text{Composite}(n) \iff [n > 1 \wedge \sim \text{Prime}(n)].$$

More directly, we could write

$$\text{Composite}(n) \iff [n > 1 \wedge (\exists m)(m > 1 \wedge m \mid n)].$$