Proofs I

1.5.5: A circle has center $(2,4)$.

(a) Prove that $(-1,5)$ and $(5,1)$ are not both on the circle.

(b) Prove that if the radius of the circle is less than 5, then the circle does not intersect the line $y = x - 6$.

(c) Prove that if $(0,3)$ is not inside the circle, then $(3,1)$ is not inside the circle.

(a) Since the circle has center $(2,4)$, it consists of those points (x,y) that satisfy the equation

$$(x-2)^2 + (y-4)^2 = R^2$$

for some value R, the radius (by the definition of a circle). If $(-1,5)$ is on the circle, then $R^2 = (-1-2)^2 + (5-4)^2 = 10$. If $(5,1)$ is on the circle, then $R^2 = (5-2)^2 + (1-4)^2 = 18$. As $10 \neq 18$, it cannot be the case that both points lie on the circle.

(b) Suppose that the point $(x_0, y_0) \in \mathbb{R}^2$ lies on both the circle and the line $y = x - 6$ (this can be so if and only if they intersect). Then in particular, we have $y_0 = x_0 - 6$, and $(x_0-2)^2 + (y_0-4)^2 = R^2$. Using the first equality gives

$$(x_0-2)^2 + (x_0-10)^2 = 2x_0^2 - 24x_0 + 104 = R^2.$$

Thus, we have from the quadratic formula

$$x_0 = 6 \pm \sqrt{-16 + \frac{1}{2}R^2}.$$

For this to have a real solution, we require $-16 + \frac{1}{2}R^2 \geq 0$, or

$$R^2 \geq 32.$$

Were $R < 5$, then $R^2 < 25 < 32$, a contradiction.

(c) If $(0,3)$ is not inside the circle, then its distance from the center is

$$(0-2)^2 + (3-4)^2 = 5 > R^2.$$

We compute the distance of $(3,1)$ from the center as

$$(3-2)^2 + (1-4)^2 = 10 > 5 > R^2,$$

so $(3,1)$ is also not inside the circle.

1.5.7: Suppose a, b, c, and d are positive integers. Prove each biconditional statement.

(a) ac divides bc if and only if a divides b.

(b) $a+1$ divides b and b divides $b+3$ if and only if $a=2$ and $b=3$

(c) $a+c=b$ and $2b-a=d$ if and only if $a=b-c$ and $b+c=d$.

(d) $a+2c\neq d$ or $b-a\neq 2d$ if and only if $b+2c\neq 3d$ or $3a+4c\neq b$.

(a) First, suppose $b=ak$ for some positive integer k. Then $bc=akc=(ac)k$; that is, $ac \mid bc$.

Now suppose $bc=(ac)k$ for some positive integer k. As $c>0$, we may divide both sides by c (or, more formally, apply the Cancellative Law for positive integers) to get $b=ak$; that is, $a \mid b$.

(b) Suppose $a=2$ and $b=3$. Then $a+1=3=1b$, so $a+1 \mid b$, and $b+3=6=2b$, so $b \mid b+3$.

Now suppose $b=(a+1)k$ and $b+3=b\ell$, for positive integers k and ℓ. From the second equation, we have $3=b(\ell-1)$; as 3 is prime, this can only be the case if $b=1$ or $b=3$. In the former case, we would have $1=(a+1)k$, forcing $a+1=1=k$, forcing $a=0$, contradicting positivity. Thus, we must have $b=3$, hence $3=(a+1)k$, hence $a+1=1$ or $a+1=3$. We have seen the first causes a contradiction, so it must be that $a+1=3$, or $a=2$.

(c) Suppose $a=b-c$ and $b+c=d$. Then $a+c=b-c+c=b$, and $2b-a=2b-(b-c)=b+c=d$.

Now suppose $a+c=b$ and $2b-a=d$. Then $a=a+c-c=b-c$, and $b+c=2b-a+a+c-b=d+b-b=d$.

(d) By taking the contrapositives (and the fact that $\sim P \iff \sim Q$ is equivalent to $P \iff Q$), we get that the given statement is equivalent to "$a+2c=d$ and $b-a=2d$ if and only if $b+2c=3d$ and $3a+4c=b$".

Suppose the left hand statements are true. Then $b+2c=b-a+a+2c=d+2d=3d$, and $3a+4c=a+2(a+2c)=a+2d=a+b-a=b$.

Now suppose the right hand statements are true. Then $a+2c=a+b+2c-b=a+3d-3a-4c=-2(a+2c)+3d$; adding $2(a+2c)$ to both sides and applying the Cancellative Law gives $a+2c=d$. Further, $b-a=b+2c-2c-a=3d-d=2d$ (using that result).