Proofs II

1.6.3: Prove that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

If $n > 5$ is odd, then $n - 3$ is even and greater than 2; suppose there exist primes p and q such that $n - 3 = p + q$. Then $n = 3 + p + q$, the sum of three primes.

1.6.4 Provide either a proof or a counterexample for each of these statements.

(a) For all positive integers x, $x^2 + x + 41$ is a prime.

(b) In the universe of all reals, $(\forall x)(\exists y)(x + y = 0)$.

(c) In the universe of all reals, $(\forall x)(\forall y)(x > 1 \wedge y > 0 \implies y^x > x)$.

(d) For integers a, b, c, if a divides bc, then either a divides b, or a divides c.

(a) This is false: for $x = 41$, we have $x^2 + x + 41 = 41 \cdot 43$.

(b) This is true: given $x \in \mathbb{R}$, choosing $y = -x \in \mathbb{R}$ gives $x + y = x - x = 0$.

(c) This is false: take $x = 2 > 1$ and $y = 1 > 0$. Then $y^x = 1^2 = 1 \leq 2 = x$.

(d) This is false: take $a = 6$, $b = 2$ and $c = 3$. Then a divides bc (with remainder 1), but neither b nor c.

1.7.3: Prove that

(a) $5n^2 + 3n + 4$ is even, for all integers n.

(b) for all integers n, if $5n + 1$ is even, then $2n^2 + 3n + 4$ is odd.

(c) the sum of five consecutive integers is always dividible by 5.

(d) $n^3 - n$ is divisible by 6, for all intgers n.

(a) If n is even, then n^2 is even, so $5n^2$, $3n$, and 4 are all even; thus, their sum is even. If n is odd, then n^2 is odd, so $5n^2$ and $3n$ are odd while 4 is even; the sum of two odd integers is even, and the sum of two evens is even, so the total is even as well.

(b) If $5n + 1$ is even, then $5n$ is odd, so n must be odd. Then $3n$ is odd, while $2n^2$ and 4 are even, so the sum is of one odd term and two even ones, hence it is odd.

(c) Suppose the integers are n, $n+1$, $n+2$, $n+3$, and $n+4$. Then their sum is $5n + 10 = 5(n+2)$, which is divisible by 5.

(d) We show separately that $n^3 - n$ is divisible by 2 and by 3, which implies it is divisible by $6 = 2 \cdot 3$ because they share no common factors. If n is even, then n^3 is even, and $n^3 - n$ is the difference of two even numbers, hence even. If n is odd, then n^3 is odd, and $n^3 - n$ is the difference of two odd numbers, hence even. In both cases, $2 \mid n^3 - n$.

We note $n^3 - n = n(n^2 - 1)$. If $3 \mid n$, then $3 \mid n(n^2 - 1)$. If $n = 3k + 1$, then $n^2 = 9k^2 + 6k + 1$, so $n^2 - 1 = 3k(3k + 2)$, and $3 \mid n(n^2 - 1)$. If $n = 3k + 2$, then $n^2 = 9k^2 + 12k + 4$, so $n^2 - 1 = 3(3k^2 + 6k + 1)$, and $3 \mid n(n^2 - 1)$.