Sets II

2.2.6: Give an example of nonempty sets A, B, and C such that:

(a) $C \subseteq A \cup B$ and $A \cap B \nsubseteq C$

(b) $A \subseteq B$ and $C \subseteq A \cap B$

(c) $A \cup B \subseteq C$ and $C \nsubseteq B$

(d) $A \nsubseteq B \cup C$, $B \nsubseteq A \cup C$, and $C \subseteq A \cup B$

(a) Here, we can put $A = \{a, b\}$, $B = \{b, c\}$, and $C = \{a, c\}$.

(b) Here, we can put $A = \{a, c\}$, $B = \{a, b, c\}$, and $C = \{c\}$. It's worth noting that, if $A \subseteq B$, then $A \cap B = A$, so the second requirement is $C \subseteq A$.

(c) Here, we can put $A = \{a\}$, $B = \{b\}$, and $C = \{a, b\}$.

(d) Here, we can put $A = \{a, b\}$, $B = \{c, d\}$, and $C = \{a, c\}$. Then $b \notin B \cup C$ and $d \notin A \cup C$, but $C \subseteq A \cup B$.

2.2.11: Provide counterexamples for each of the following:

(a) If $A \cup C \subseteq B \cup C$, then $A \subseteq B$

(b) If $A \cap C \subseteq B \cap C$, then $A \subseteq B$

(c) If $(A - B) \cap (A - C) = \emptyset$, then $B \cap C = \emptyset$.

(a) Suppose $A = \{a\}$, $B = \{b\}$, and $C = \{a, b, c\}$; then $A \cup C = B \cup C = C$, but $A \nsubseteq B$ (in fact, A and B are disjoint!).

(b) Suppose $A = \{a\}$, $B = \{b\}$ and $C = \{c\}$; then $A \cap C = B \cap C = A \cap B = \emptyset$, so in particular $A \nsubseteq B$.

(c) Suppose $A = \{b, c\}$, $B = \{a, c\}$, and $C = \{a, b\}$; then $A - B = \{b\}$ and $A - C = \{c\}$, which are disjoint, but $B \cap C = \{a\} \neq \emptyset$.

2.3.7: Let $\mathscr{A} = \{A_\alpha : \alpha \in \Delta\}$ be a family of sets, and let B be a set. Prove that

$$B \cap \bigcup_{\alpha\in\Delta} A_\alpha = \bigcup_{\alpha\in\Delta} (B \cap A_\alpha).$$

An element x is in the set $B \cap \bigcup_{\alpha\in\Delta} A_\alpha$ if $x \in B$ and there exists some (not necessarily unique) $\xi \in \Delta$ such that $x \in A_\xi$. This can happen if and only if there exists some (not necessarily unique) $\xi \in \Delta$ such that $x \in B \cap A_\xi$ (because the choice of ξ is independent of the truth value of $x \in B$—we can "move the quantifier outside"), which is exactly the condition that x is in the set $\bigcup_{\alpha\in\Delta}(B \cap A_\alpha)$. Since they have logically equivalent membership conditions, the sets are equal.

To argue another way, if $x \in B$ and $x \in A_\xi$, then $x \in B \cap A_\xi$, so the left is a subset of the right, and if $x \in B \cap A_\xi$, then $x \in B$ and $x \in A_\xi$, so the right is a subset of the left, also establishing equality.