Sets III

2.1.18: Let A and B be sets. Prove that

(a) $A = B$ if and only if $\mathscr{P}(A) = \mathscr{P}(B)$.

(b) if A is a proper subset of B, then $\mathscr{P}(A)$ is a proper subset of $\mathscr{P}(B)$.

(a) If $A = B$, then clearly $\mathscr{P}(A) = \mathscr{P}(B)$. Now suppose $\mathscr{P}(A) = \mathscr{P}(B)$. That is, we are guaranteed that if $S \subseteq A$, then $S \subseteq B$. Suppose $a \in A$; then $\{a\} \subseteq A$, so $\{a\} \subseteq B$. This implies that $a \in B$. Similarly, if $b \in B$, then $\{b\} \subseteq B$, so $\{b\} \subseteq A$, so $b \in A$. Thus, $A = B$.

(b) Suppose $b \in B - A$ (this must happen if $A \subsetneq B$). We know $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. Also, $\{b\} \in \mathscr{P}(B)$. However, $\{b\} \notin \mathscr{P}(A)$, as $b \notin A$ by construction. Thus, $\mathscr{P}(A) \subsetneq \mathscr{P}(B)$.

2.3.14: Let $\mathscr{A}$ be a family of pairwise disjoint sets. Prove that if $\mathscr{B} \subseteq \mathscr{A}$, then $\mathscr{B}$ must be a pairwise disjoint family of sets.

Suppose $A, B \in \mathscr{B}$ are sets. Then $A, B \in \mathscr{A}$ as well; since the elements of $\mathscr{A}$ are pairwise disjoint, either $A = B$ or $A \cap B = \emptyset$. Since A and B were arbitrary elements of $\mathscr{B}$, this shows $\mathscr{B}$ is pairwise disjoint as well.

2.3.17: Suppose $\mathscr{A} = \{A_i : i \in \mathbb{N}\}$ is a family of sets such that for all $i, j \in \mathbb{N}$, if $i \leq j$, then $A_j \subseteq A_i$. (Such a family is called a *nested family* of sets.)

(a) Prove that for every $k \in \mathbb{N}$, $\bigcap_{i=1}^{k} A_i = A_k$.

(a) We certainly have that $\bigcap_{i=1}^{k} A_i \subseteq A_k$, as it is one of the members of the intersection. Now suppose $a \in A_k$, and consider $j \leq k$. Then by the nested condition, we have $A_k \subseteq A_j$, so in particular $a \in A_j$. This is true for all $j \leq k$, so $a \in \bigcap_{i=1}^{k} A_i$; since a was arbitrary, $A_k \subseteq \bigcap_{i=1}^{k} A_i$. Thus, the two sets are equal.

2.4.5: Use the PMI to prove the following for all natural numbers:

(a) $n^3 + 5n + 6$ is divisible by 3

(b) $4^n - 1$ is divisible by 3

(c) $n^3 - n$ is divisible by 6

(d) $(n^3 - n)(n + 2)$ is divisible by 12

(a) For $n = 1$, the given expression is 12, which 3 divides. Suppose $3 \mid n^3 + 5n + 6$. Then

$$(n+1)^3 + 5(n+1) + 6 = (n^3 + 5n + 6) + 3n^2 + 3n + 6,$$

a sum of terms each divisible by 3. Thus, the statement is true for all n.

(b) For $n = 1$, the given expression is 3, which 3 divides. Suppose $3 \mid 4^n - 1$. Then

$$4^{n+1} - 1 = 4(4^n - 1) + 3,$$

a sum of terms each divisible by 3. Thus, the statement is true for all n.

(c) For $n = 1$, the given expression is 0, which 6 (and all other integers) divides. Suppose $n^3 - n = 6k$. Then

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 2n = (n^3 - n) + 3n(n-1).$$

Now, $n(n-1) = 2j$ for some j (a previous result), so

$$(n+1)^3 - (n+1) = 6(k + j).$$

Thus, the statement is true for all n.

(d) For $n = 1$, the given expression is 0, which 12 divides. Suppose $(n^3 - n)(n + 2) = 12k$. Then

$$([n+1]^3 - [n+1])([n+1]+2) = n^4 + 3n^3 + 2n^2 + 3n^3 + 9n^2 + 6n = n^4 + 6n^3 + 11n^2 + 6n = (n^3 - n)(n+2) + 4n(n+1)(n+2).$$

Now, one of n, $n+1$, and $n+2$ is divisible by three. Thus (and this might merit a more careful proof), $4n(n+1)(n+2) = 12j$, so

$$([n+1]^3 - [n+1])([n+1] + 2) = 12(k + j).$$

The given statement is therefore true for all n.