Induction

2.5.7: Use the Principle of Complete Induction to prove the following properties of the Fibonacci numbers:

(a) f_n is a natural number for all natural numbers n.

(b) $f_{n+6} = 4f_{n+3} + f_n$ for all natural numbers n.

(c) For every natural number a, $f_a f_n + f_{a+1} f_{n+1} = f_{a+n+1}$ for all natural numbers n.

(a) We know $f_1 = f_2 = 1 \in \mathbb{N}$; consider $n > 2$. Suppose $f_m \in \mathbb{N}$ for all natural numbers $m < n$. In particular, f_{n-1} and f_{n-2} are natural. Since $f_n = f_{n-1} + f_{n-2}$ and the sum of natural numbers is natural, $f_n \in \mathbb{N}$. By the PCI, this is therefore true for all $n \in \mathbb{N}$.

(b) We know $13 = f_7 = 4f_4 + f_1 = 4(3) + 1$ and $21 = f_8 = 4f_5 + f_2 = 4(5) + 1$. Suppose $f_{m+6} = 4f_{m+3} + f_m$ for all $m < n$. Then we compute for $n > 2$

$$\begin{aligned} f_{n+6} &= f_{n+5} + f_{n+4} \\ &= 4f_{(n-1)+3} + f_{n-1} + 4f_{(n-2)+3} + f_{n-2} \\ &= 4\left(f_{n+2} + f_{n+1}\right) + \left(f_{n-1} + f_{n-2}\right) \\ &= 4f_{n+3} + f_n. \end{aligned}$$

So, the statement is true for all $n \in \mathbb{N}$.

2.6.13: Prove the number of permutations of a subcollection of r objects from a larger collection of n objects is $\frac{n!}{(n-r)!}$

(b) by induction on n.

(b) If $n = 1$, then the only choice for r is also 1, and there is 1 permutation because there is one possible order, and $1 = \frac{1!}{0!}$. Now suppose the number of permutations of r objects from n is $\frac{n!}{(n-r)!}$ for $1 \le r \le n$. If we had $n+1$ objects and wished to construct a permutation of r of them (for $1 \le r \le n+1$), we could do so by first choosing one of the $n+1$ objects to be first, then constructing a permutation of $r-1$ of the remaining n objects (and $0 \le r \le n$). If $r = 1$, we're done after the first choice, and had $n+1 = \frac{(n+1)!}{(n+1-1)!}$. Otherwise, the number of options is

$$(n+1)\frac{n!}{(n-(r-1))!} = \frac{(n+1)!}{((n+1)-r)!}.$$

By the Principle of Mathematical Induction, this formula therefore holds for all $n \in \mathbb{N}$.

2.6.24: The nth *pyramid number*, p_n, is the number of balls of equal diameter that can be stacked in a pyramid whose base is an n by n square. The first few pyramid numbers are $p_1 = 1$, $p_2 = 5$, $p_3 = 14$, and $p_4 = 30$. Show that

(a) $p_n = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every natural number n.

(b) $p_n = \binom{n+2}{3} + \binom{n+1}{3}$ for $n \geq 2$.

(a) We will take the first equality more or less for granted: to construct a pyramid on a base of length n, we first lay down n^2 balls, then construct a pyramid of base length $n-1$ on top by putting a ball between every square formed by four adjacent balls in the base. Now, clearly $1^2 = 1 = \frac{1(2)(3)}{6}$. Suppose we know $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+2)(2n+1)}{6}$. Then

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+2)(2n+1)}{6} + (n+1)^2 \\ &= \frac{2n^3 + 9n^2 + 13n + 6}{6} \\ &= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6}. \end{aligned}$$

(b) We prove first that $\binom{n+1}{3} + (n+1)^2 = \binom{n+3}{3}$. This may be done purely algebraically:

$$\binom{n+1}{3} + (n+1)^2 = \frac{(n+1)n(n-1)}{6} + (n+1)^2 = \frac{n+1}{6}\left(n^2 + 5n + 6\right) = \frac{(n+1)(n+2)(n+3)}{6} = \binom{n+3}{3}.$$

We could also construct a (contrived) combinatorial task to prove this: we must either produce a set of three objects from $n+1$, or paint a black and a white mark on two objects, which may coincide. The left hand expression counts this directly: there are $\binom{n+1}{3}$ ways to produce the set, and $(n+1)$ choices for the black mark times $(n+1)$ for the white. The right hand side counts indirectly: we add two "placeholder" objects p and q to the collection and choose three from the augmented set; if p is chosen, we mark the chosen objects such that the smaller one chosen gets the black mark, if q is chosen similarly but the smaller gets the white mark. This represents the same situation: if neither p nor q is chosen we have a set of three objects, if exactly one is chosen we have two distinct objects and a consistent way of determining which has which mark, and if both are chosen we have a single object which receives both marks.

With either proof, we have a useful lemma. Now, if $n = 2$, then $p_2 = 5 = \binom{4}{3} + \binom{3}{3} = 4 + 1$. Assume p_n has the given form, and consider

$$p_{n+1} = p_n + (n+1)^2 = \binom{n+2}{3} + \binom{n+1}{3} + (n+1)^2 = \binom{n+3}{3} + \binom{n+2}{3}.$$

Thus, by the Principle of Mathematical Induction, p_n has the given form for all $n \in \mathbb{N}$.